Coherence decay of wavepackets in turbulent jets by stochastic modelling under location uncertainty

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Joint work with André Cavalieri[2], Tim Colonius[3], Peter Jordan[4] and Étienne Mémin[1]

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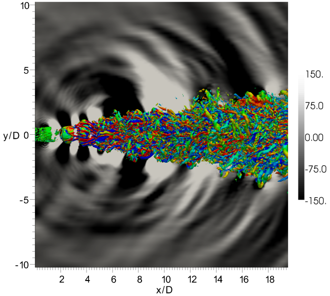
Motivation: the jet noise problem

São Paulo airport.



G. Daviller (2011).

- Jet noise dominant during take off.
- Becomes limiting for certifications.
- Noise comes from the flow.

Motivation: the jet noise problem

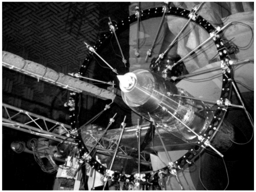
Wavepackets

Jordan & Colonius (2013)
Cavalieri et al. (2012,2013)
Tinney et al. (2008)

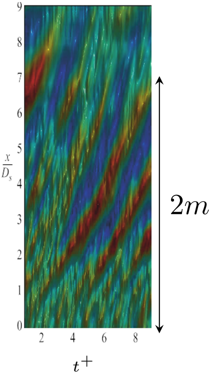
Where does the noise come from?

- Turbulent stochastic eddies?
- Or something more organised?

- Wavepackets in pressure/velocity field.



Tinney & Jordan 2008; Co-axial transonic heated jet $Re = 5 \times 10^6$.



Near-field pressure.

- Acoustic directivity as extended source *(low azimuthal angles)*.

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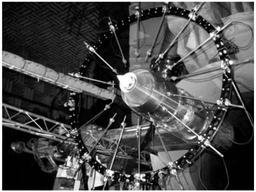

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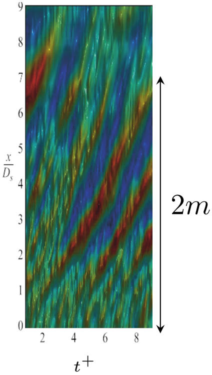


Near-field pressure.

- Acoustic directivity as extended source *(low azimuthal angles)*.

Source: likely a wavepacket shape.

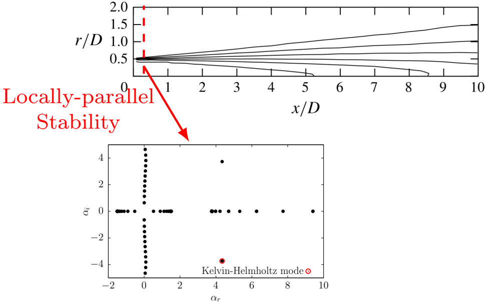
Motivation: the jet noise problem

Linear models

Jordan & Colonius (2013)
Cavalieri et al. (2013)
Sinha et al. (2014)
Baqui et al. (2015)

Wavepackets: Propagated linear instability waves

Experimental subsonic jet mean flow.



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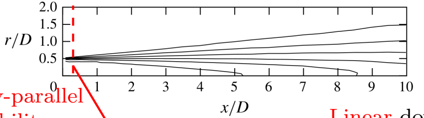
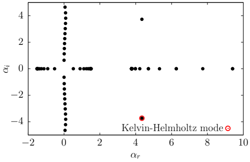


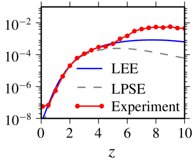
Locally-parallel Stability



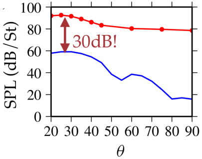
Linear downstream propagation of the Kelvin-Helmholtz mode.

Linearised Euler equations

Centerline $|u|^2$:



Far-field sound:

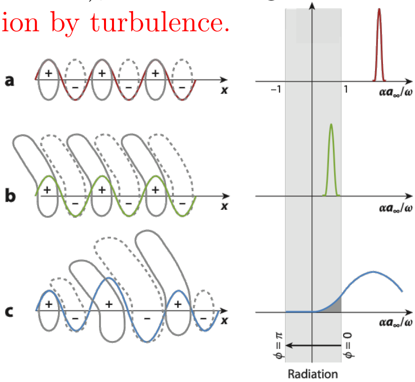


Non-linearities important for far-field prediction.

Motivation: the jet noise problem

What is missing?

- Propagation condition: $c = \frac{\omega}{|\boldsymbol{k}|}$, $k_y = \sqrt{\frac{\omega^2}{c^2} - k_x^2}$
  $\Rightarrow$ supersonic axial phase velocity.
- Wavepacket envelop shape allows propagation (due to convolution), but not enough.
- Intensification by turbulence.



From Jordan & Colonius (2013)

Motivation: the jet noise problem

Cavalieri et al. (2011, 2013)
Baqui et al. (2015)

Several representations for intensifying phenomena

- Non-linear term.
- Jittering wavepacket.
- Coherence (2-points statistics).

Motivation: the jet noise problem

The missing piece

Cavalieri et al. (2011, 2013)
Baqui et al. (2015)

Several representations for intensifying phenomena

- Non-linear term.
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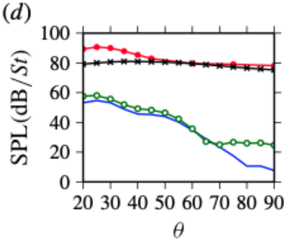
Coherence-matched wavepackets: right acoustic emission



From Baqui et al.(2015)

Motivation: the jet noise problem

Objective:

Simplified model for wavepackets (or coherent structures) evolving within turbulent flows.

- Model-based.
- Predict coherence decay.
- Stochastic modelling under location uncertainty.

Outline

Classical methods

Overview

Lumley (1967)
Herbert (1997)
McKeon et al. (2010)
Schmid (2010)
Towne et al. (2015,2018)

Extract from data:

- Proper Orthogonal Decomposition (POD).
- Dynamic Mode Decomposition (DMD), and variants. . .
- Spectral Proper Orthogonal Decomposition (SPOD).

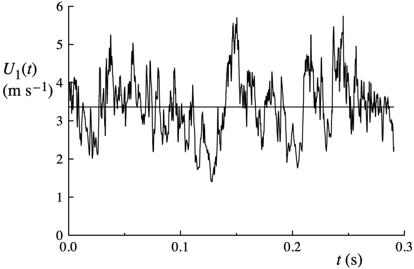
Models:

- Linear stability.
- PSE, OWNS.
- Resolvent analysis.

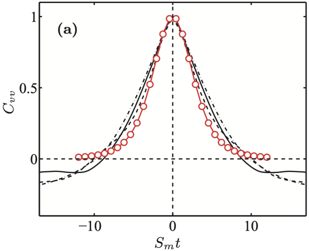
Classical methods

SPOD

Turbulent signal:



(From Pope 2000).



(From Jimenez 2013).

$$\boldsymbol{C}_{ij}(\boldsymbol{x},\boldsymbol{x}',\theta,\tau) = \langle u_i(\boldsymbol{x},\theta_0,t)u_j(\boldsymbol{x}',\theta_0+\theta,t+\tau)\rangle \qquad \boldsymbol{x} = (x,r)^T$$

Fourier transform in (θ, t)

$$\boldsymbol{S}^{ij}_{m,\omega}(\boldsymbol{x},\boldsymbol{x}') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}\boldsymbol{C}_{ij}(\boldsymbol{x},\boldsymbol{x}',\theta,\tau)e^{i(\omega\tau-2\pi m\theta)}\mathrm{d}\tau\mathrm{d}\theta,$$

Classical methods

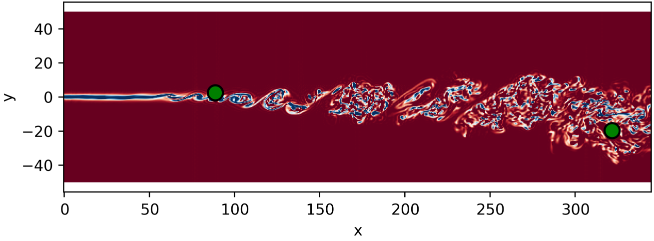
SPOD

Some quantities:

- Cross power spectral density tensor (CSD): $\boldsymbol{S}^{ij}_{m,\omega}(\boldsymbol{x},\boldsymbol{x}')$.
- Power spectral density (PSD): $\boldsymbol{S}^{ii}_{m,\omega}(\boldsymbol{x},\boldsymbol{x})$.
- Coherence: $\gamma^2_{m,\omega,i,j}(\boldsymbol{x},\boldsymbol{x}') = \frac{|\boldsymbol{S}^{ij}_{m,\omega}(\boldsymbol{x},\boldsymbol{x}')|^2}{\boldsymbol{S}^{ii}_{m,\omega}(\boldsymbol{x},\boldsymbol{x})\boldsymbol{S}^{jj}_{m,\omega}(\boldsymbol{x}',\boldsymbol{x}')}$

  $\gamma = 1$: Perfectly syncronised.

  $\gamma = 0$: Decorrelated.



Mixing layer $Re = 300$, *Spanwise vorticity* ξ.

Classical methods

SPOD

Towne et al. (2018)

Spectral proper orthogonal decomposition (SPOD):

$$\boldsymbol{S}_{m,\omega}\boldsymbol{W}\boldsymbol{\Phi}^{\mathrm{SPOD}}_{k,m,\omega} = \lambda_{k,m,\omega}\boldsymbol{\Phi}^{\mathrm{SPOD}}_{k,m,\omega},$$

Interest:

- Perfectly coherent modes.
- Decorrelated from each other.
- Orthonormal bases.
- Sorted by energy content.

Classical methods

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Interest:

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Low order representation:

$$\boldsymbol{S}_{m,\omega}(\boldsymbol{x},\boldsymbol{x}') \approx \sum_{k=1}^{N} \boldsymbol{\Phi}^{\text{SPOD}}_{k,m,\omega}(\boldsymbol{x})\lambda_{k,m,\omega}\boldsymbol{\Phi}^{*,\text{SPOD}}_{k,m,\omega}(\boldsymbol{x}')$$

And then access to γ^2.

Classical methods

Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \dfrac{1}{Re}\Delta \boldsymbol{u} \\ \nabla \cdot \boldsymbol{u} = 0. \end{cases}$$

Classical methods

Resolvent analysis

Navier-Stokes equations:

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Linearisation: Around the mean flow $\bar{\boldsymbol{u}} = (U(y), 0, 0)^T$.

$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}'$

$$\begin{cases} \dfrac{\partial \boldsymbol{u}'}{\partial t} + (\bar{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u}' + (\boldsymbol{u}' \cdot \nabla)\bar{\boldsymbol{u}} + \nabla p' - \dfrac{1}{Re}\Delta \boldsymbol{u}' = -(\boldsymbol{u}' \cdot \nabla)\boldsymbol{u}' \\ \nabla \cdot \boldsymbol{u}' = 0. \end{cases}$$

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Fourier transform:

$$\boldsymbol{u}'(x, r, \theta, t) = \frac{1}{2\pi} \sum_m \int_{-\infty}^{\infty} \hat{\boldsymbol{u}}_{m,\omega}(x, r)\, e^{i(-\omega t + 2\pi m\theta)}\, \mathrm{d}\omega.$$

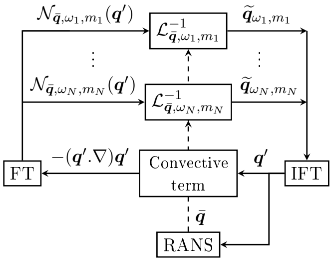
Classical methods

Schmid & Henningson (2001)
McKeon & Sharma (2010,2013)
Towne et al. (2020)
Morra et al. (2019,2020)
Nogueira et al. (2020)
Martini et al. (2020)

Non-linearity as an "external forcing"

Navier-Stokes in the frequency-wavenumber domain:

$$\left(A_{\bar{\boldsymbol{u}},m,\omega} - i\omega E\right) \hat{\boldsymbol{q}}_{m,\omega} = \mathcal{N}_{\bar{\boldsymbol{u}},m,\omega}(\boldsymbol{q}') = \mathcal{F}(-\boldsymbol{u}' \cdot \nabla \boldsymbol{u}').$$



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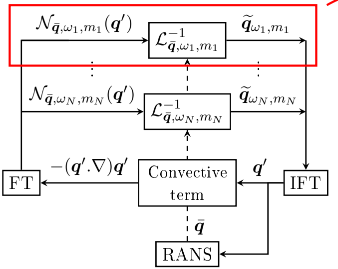

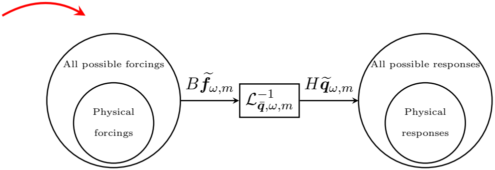

Identify relevant non-linearities.

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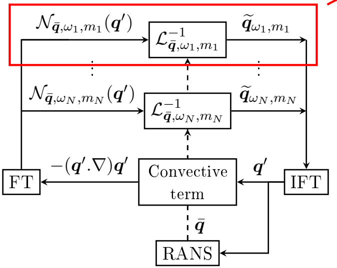

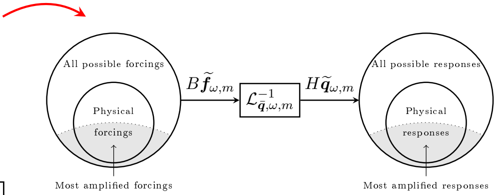

- Resolvent analysis.

Identify relevant non-linearities.

Classical methods

Linearised problem with external forcing:

$$H\hat{\boldsymbol{q}}_{m,\omega} = \underbrace{H(A_{\bar{\boldsymbol{u}},m,\omega} - i\omega E)^{-1}B}_{\text{SVD}}\,\hat{\boldsymbol{f}}_{m,\omega}.$$

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SVD to maximise Rayleigh quotient:

$$\max_{\hat{\boldsymbol{f}}_{m,\omega}} \frac{\|H\hat{\boldsymbol{q}}_{m,\omega}\|^2}{\|\hat{\boldsymbol{f}}_{m,\omega}\|^2} = \frac{\left\|\left(H(A_{\bar{\boldsymbol{u}},m,\omega} - i\omega E)^{-1}B\right)\hat{\boldsymbol{f}}_{m,\omega}\right\|^2}{\|\hat{\boldsymbol{f}}_{m,\omega}\|^2}.$$

Classical methods

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Most amplified harmonic forcing/response modes:

$$H(A_{\bar{\boldsymbol{u}},m,\omega} - i\omega E)^{-1}B = U\Sigma V^*$$

$$H(A_{\bar{\boldsymbol{u}},m,\omega} - i\omega E)^{-1}B\boldsymbol{V}_i = \sigma_i\boldsymbol{U}_i$$

with $U = (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_N)$, $V = (\boldsymbol{V}_1, \ldots, \boldsymbol{V}_N)$ and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_N)$.

Classical methods

Resolvent analysis

Towne et al. (2017)
Cavalieri et al. (2019)

Link with SPOD:

If $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ (*i.e.* $\boldsymbol{f}$ in the data) is a Gaussian white noise, then SPOD and resolvent modes match.

Proof:

$\mathcal{R} = H(A_{\bar{\boldsymbol{u}},m,\omega} - i\omega E)^{-1}B$

$$\boldsymbol{S}_{m,\omega} = \mathbb{E}\left(\hat{\boldsymbol{q}}_{m,\omega}\hat{\boldsymbol{q}}^*_{m,\omega}\right) = \mathcal{R}\,\underbrace{\mathbb{E}\left(\hat{\boldsymbol{f}}_{m,\omega}\hat{\boldsymbol{f}}^*_{m,\omega}\right)}_{\mathbb{I}}\mathcal{R}^*$$

Then, eigenfunctions of $\boldsymbol{S}_{m,\omega}$ (SPOD) are resolvent modes.

Classical methods

Adding eddy viscosity

Reynolds & Hussain (1972)
Morra et al. (2019)
Symon et al. (2020)

Eddy viscosity

Triple decomposition

$$\boldsymbol{u}(\boldsymbol{x},t) = \underbrace{\bar{\boldsymbol{u}}(\boldsymbol{x})}_{\text{average}} + \underbrace{\widetilde{\boldsymbol{u}}(\boldsymbol{x},t)}_{\text{coherent}} + \underbrace{\boldsymbol{u}'(\boldsymbol{x},t)}_{\text{turbulent}}$$

Ensemble/phase averaging $\langle \boldsymbol{u} \rangle = \bar{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$.

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We obtain generalised Reynolds stresses

$$\underbrace{\mathcal{F}\left(\overline{(\widetilde{\boldsymbol{u}}\cdot\nabla)\,\widetilde{\boldsymbol{u}}} - (\widetilde{\boldsymbol{u}}\cdot\nabla)\,\widetilde{\boldsymbol{u}}\right)}_{\hat{\boldsymbol{f}}} + \underbrace{\mathcal{F}\left(\overline{(\boldsymbol{u}'\cdot\nabla)\,\boldsymbol{u}'} - (\boldsymbol{u}'\cdot\nabla)\,\boldsymbol{u}'\right)}_{\approx\nabla\cdot(\nu_t\nabla\widetilde{\boldsymbol{u}})}$$

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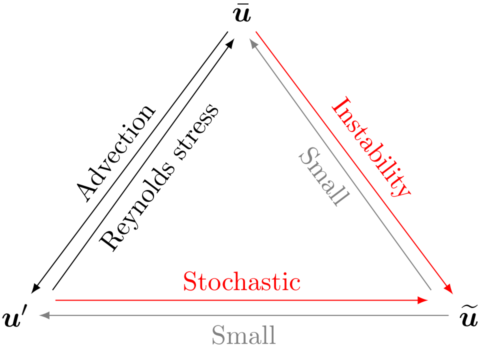
- Models energy transfers from scales with high production.
- $\nu_t(\omega = 0) \neq \nu_t(\omega \neq 0)$.
- Purely dissipative $\Rightarrow$ forbid backscatter.

Stochastic model

Main idea

Decomposition: $\boldsymbol{u}(\boldsymbol{x},t) = \underbrace{\bar{\boldsymbol{u}}(\boldsymbol{x})}_{\text{average}} + \underbrace{\widetilde{\boldsymbol{u}}(\boldsymbol{x},t)}_{\text{coherent}} + \underbrace{\boldsymbol{u}'(\boldsymbol{x},t)}_{\text{turbulent}}$



G. Tissot, A. Cavalieri, & E. Mémin Input-output analysis of the stochastic Navier Stokes equations: Application to turbulent channel flow *Phys. Rev. Fluids (2023)*

G. Tissot, A. Cavalieri, & E. Mémin Stochastic linear modes in a turbulent channel flow, *J. Fluid Mech. (2021)*

Stochastic model

Modelling under location uncertainty

Particle displacement

$$\boldsymbol{X}(\boldsymbol{x},t) = \boldsymbol{X}(\boldsymbol{x},t_0) + \int_{t_0}^{t} \boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}t + \underbrace{\int_{t_0}^{t} \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t}_{\text{Brownian motion}} \quad ,$$

Differential form

$$\mathrm{d}\boldsymbol{X}(\boldsymbol{x},t) = \underbrace{\boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}t}_{\text{Resolved, smooth}} + \underbrace{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t}_{\text{Unresolved, random}}$$

with

$$\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t = \int_\Omega \check{\boldsymbol{\sigma}}(\boldsymbol{x},\boldsymbol{x}',t)\mathrm{d}\boldsymbol{B}_t(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'.$$

Stochastic model

Modelling under location uncertainty

Kunita (1997)
Mémin (2014)
Resseguier et al. (2017)

Stochastic transport operator

$$\mathrm{d}_t\theta + \nabla\cdot(\theta\boldsymbol{u}^\star)\mathrm{d}t + \nabla\cdot(\theta\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t) = \frac{1}{2}\nabla\cdot(\boldsymbol{a}\nabla\theta)\,\mathrm{d}t,$$

with the *drift velocity*

$$\boldsymbol{u}^\star = \boldsymbol{u} - \frac{1}{2}\nabla\cdot\boldsymbol{a} + \boldsymbol{\sigma}(\nabla\cdot\boldsymbol{\sigma}).$$

and $\boldsymbol{a}_{ij}(\boldsymbol{x},t) = \int_\Omega \boldsymbol{\sigma}^{ik}(\boldsymbol{x},\boldsymbol{y},t)\boldsymbol{\sigma}^{kj}(\boldsymbol{x},\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T.$

Stochastic model

Kunita (1997)
Mémin (2014)
Resseguier et al. (2017)
Chandramouli et al. (2018)
Li et al. (2023)

Stochastic Navier-Stokes equations

$$\begin{cases} \mathrm{d}_t \boldsymbol{u} + (\boldsymbol{u}^\star \cdot \nabla)\, \boldsymbol{u}\, \mathrm{d}t + (\boldsymbol{\sigma} \mathrm{d}\boldsymbol{B}_t \cdot \nabla)\, \boldsymbol{u} = -\nabla \left(p_t \, \mathrm{d}t + \mathrm{d}p_t\right) \\ \qquad + \dfrac{1}{Re} \nabla \cdot (\nabla \boldsymbol{u})\ \mathrm{d}t + \dfrac{1}{2} \nabla \cdot (\boldsymbol{a} \nabla \boldsymbol{u})\ \mathrm{d}t + \dfrac{1}{Re} \nabla \cdot (\nabla \boldsymbol{\sigma} \mathrm{d}\boldsymbol{B}_t) \\ \nabla \cdot \boldsymbol{u}^\star = 0; \quad \nabla \cdot \boldsymbol{\sigma} = 0, \\ \boldsymbol{u}^\star = \boldsymbol{u} - \dfrac{1}{2} \nabla \cdot \boldsymbol{a}. \end{cases}$$

Applications:

- Large-eddy simulations.
- Geophysical flow simulations.

Stochastic model

Linearisation $\boldsymbol{u} = \bar{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$; $\boldsymbol{u}' \approx \boldsymbol{\sigma} \mathrm{d}\boldsymbol{B}_t$

$$\begin{cases} \mathrm{d}_t \widetilde{\boldsymbol{u}} + (\bar{\boldsymbol{u}}^{\star} \cdot \nabla)\, \widetilde{\boldsymbol{u}}\, \mathrm{d}t + (\widetilde{\boldsymbol{u}} \cdot \nabla)\, \bar{\boldsymbol{u}}\, \mathrm{d}t + (\boldsymbol{\sigma} \mathrm{d}\boldsymbol{B}_t \cdot \nabla)\, \bar{\boldsymbol{u}} + (\boldsymbol{\sigma} \mathrm{d}\boldsymbol{B}_t \cdot \nabla)\, \widetilde{\boldsymbol{u}} \\ \quad = -\nabla \left(\widetilde{p}_t \,\mathrm{d}t + \mathrm{d}p_t \right) \\ \qquad + \frac{1}{Re} \nabla \cdot (\nabla \widetilde{\boldsymbol{u}})\, \mathrm{d}t + \frac{1}{2} \nabla \cdot (\boldsymbol{a} \nabla \boldsymbol{u})\, \mathrm{d}t + \frac{1}{Re} \nabla \cdot (\nabla \boldsymbol{\sigma} \mathrm{d}\boldsymbol{B}_t) \\ \nabla \cdot \widetilde{\boldsymbol{u}} = 0; \quad \nabla \cdot \boldsymbol{\sigma} = 0 \end{cases}$$

Stochastic model

Linearisation $\boldsymbol{u} = \bar{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$; $\boldsymbol{u}' \approx \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t$

$$\begin{cases} \mathrm{d}_t\widetilde{\boldsymbol{u}} + (\bar{\boldsymbol{u}}^\star \cdot \nabla)\,\widetilde{\boldsymbol{u}}\,\mathrm{d}t + (\widetilde{\boldsymbol{u}} \cdot \nabla)\,\bar{\boldsymbol{u}}\,\mathrm{d}t + (\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot \nabla)\,\bar{\boldsymbol{u}} + (\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot \nabla)\,\widetilde{\boldsymbol{u}} \\ \quad = -\nabla\left(\widetilde{p}_t\,\mathrm{d}t + \mathrm{d}p_t\right) \\ \qquad + \dfrac{1}{Re}\nabla\cdot\left(\nabla\widetilde{\boldsymbol{u}}\right)\,\mathrm{d}t + \dfrac{1}{2}\nabla\cdot\left(\boldsymbol{a}\nabla\boldsymbol{u}\right)\,\mathrm{d}t + \dfrac{1}{Re}\nabla\cdot\left(\nabla\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\right) \\ \nabla\cdot\widetilde{\boldsymbol{u}} = 0; \quad \nabla\cdot\boldsymbol{\sigma} = 0 \end{cases}$$

Ansatz

$$\begin{cases} \widetilde{\boldsymbol{u}}(x,r,\theta,t) = \hat{\boldsymbol{u}}_{m,\omega}(x,r)e^{i(2\pi m\theta - \omega t - \sum_j \omega'_j \zeta_{j,t})} \\ \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t = \mathrm{d}\boldsymbol{\xi}_{m,\omega}(x,r)e^{i(2\pi m\theta - \omega t - \sum_j \omega'_j \zeta_{j,t})} \end{cases}$$

with $\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t = \sum_{m,j} \boldsymbol{\Phi}_{B,m,j}(x,r)e^{i(2\pi m\theta)}\mathrm{d}\zeta_{j,t}$

Stochastic model

Random part

$$\sum_j i\omega'_j \hat{\boldsymbol{u}}_{m,\omega} \mathrm{d}\zeta_{j,t} = \mathcal{F}\left(\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot \nabla \widetilde{\boldsymbol{u}}\right) + \nabla q_j \mathrm{d}\zeta_{j,t},$$

Predicts the random phase... but not used here.

Stochastic model

Stochastic linear model

$$\begin{pmatrix} A_{\text{adv}}(\cdot) + \frac{\partial U}{\partial x} - D_{xx}(\cdot) & \frac{\partial U}{\partial r} - D_{xr}(\cdot) & 0 & \frac{\partial \cdot}{\partial x} \\ \frac{\partial V}{\partial x} - D_{rx}(\cdot) & A_{\text{adv}}(\cdot) + \frac{\partial V}{\partial r} - D_{rr}(\cdot) & 0 & \frac{\partial \cdot}{\partial r} \\ 0 & 0 & A_{\text{adv}}(\cdot) + \frac{1}{r} V - D_{zz}(\cdot) & im\frac{1}{r} \\ \frac{\partial \cdot}{\partial x} & \frac{1}{r}\frac{\partial r \cdot}{\partial r} & im\frac{1}{r} & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_{m,\omega} \\ \hat{v}_{m,\omega} \\ \hat{w}_{m,\omega} \\ \hat{p}_{m,\omega} \end{pmatrix}$$

$$= \begin{pmatrix} -(\dot{\boldsymbol{\xi}}_{m,\omega})_x \frac{\partial U}{\partial x} - (\dot{\boldsymbol{\xi}}_{m,\omega})_r \frac{\partial U}{\partial r} + \frac{1}{Re}\nabla \cdot (\nabla(\dot{\boldsymbol{\xi}}_{m,\omega})_x) \\ -(\dot{\boldsymbol{\xi}}_{m,\omega})_x \frac{\partial V}{\partial x} - (\dot{\boldsymbol{\xi}}_{m,\omega})_r \frac{\partial V}{\partial r} + \frac{1}{Re}\nabla \cdot (\nabla(\dot{\boldsymbol{\xi}}_{m,\omega})_y) \\ \frac{1}{Re}\nabla \cdot (\nabla(\dot{\boldsymbol{\xi}}_{m,\omega})_z) \\ 0 \end{pmatrix},$$

with $A_{\text{adv}}(\cdot) = -i\omega + U_d \frac{\partial \cdot}{\partial x} + V_d \frac{\partial \cdot}{\partial r}$,
and $D(\cdot) = \frac{1}{Re}\nabla \cdot (\nabla(\cdot)) + \frac{1}{2}\nabla \cdot (\boldsymbol{a}\nabla(\cdot))$.

Stochastic model

Stochastic linear model

In a compact form

$$\left(\boldsymbol{A}^{\mathrm{SLM}}_{\bar{\boldsymbol{u}},m,\omega} - i\omega\boldsymbol{E}\right)\hat{\boldsymbol{q}}_{m,\omega} = \boldsymbol{B}^{\mathrm{SLM}}_{\bar{\boldsymbol{u}}}\hat{\boldsymbol{f}}_{m,\omega}$$

Ensemble ⇒ Cross spectral density matrix and eigenfunctions.
Comparable to resolvent and SPOD.

Stochastic model

Stochastic linear model

In a compact form

$$\left(\boldsymbol{A}^{\mathrm{SLM}}_{\bar{\boldsymbol{u}},m,\omega} - i\omega \boldsymbol{E}\right) \hat{\boldsymbol{q}}_{m,\omega} = \boldsymbol{B}^{\mathrm{SLM}}_{\bar{\boldsymbol{u}}} \hat{\boldsymbol{f}}_{m,\omega}$$

In practice: solved rewritten as a SVD problem.

Stochastic model

Model definition

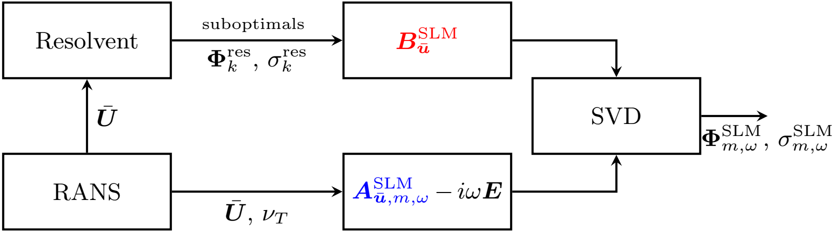


Figure: Schematic representation of the SLM procedure.

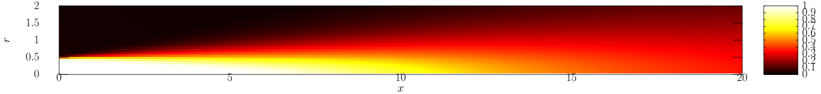
$$\dot{\boldsymbol{\xi}}_{m,\omega} = \sum_{k=2}^{N} \sigma_k^{\text{res}} \, \mathbf{\Phi}_k^{\text{res}} \underbrace{\eta_k}_{\mathcal{N}(0,1)}$$

Results

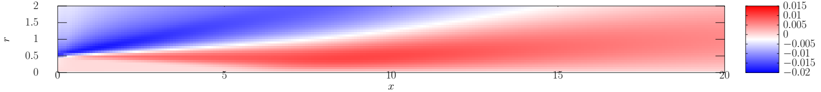
Turbulent jet

Mean flow from RANS: $M = 0.4$, $Re = 450\,000$

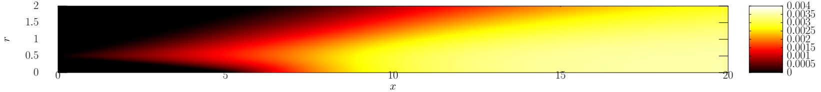
U:



V:



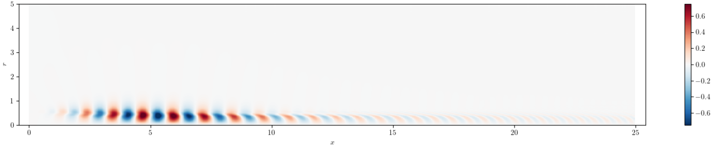
ν_T:



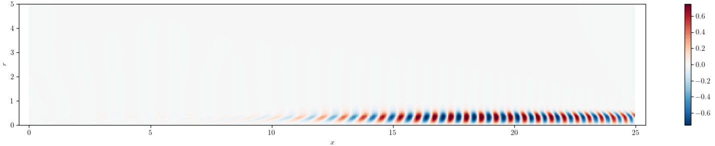
Results

Resolvent analysis: v, $St = 0.7$, focus on $m = 0$

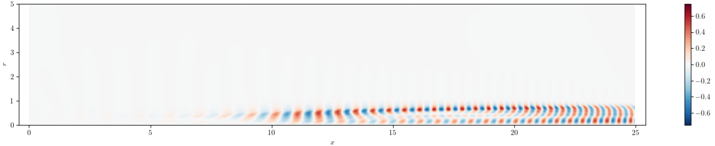
Mode 0:



Mode 1:



Mode 4:

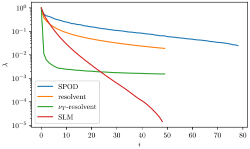


Results

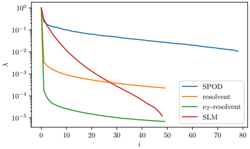
Stochastic model

Spectra

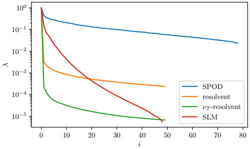
$St = 0.3$



$St = 0.6$



$St = 0.9$

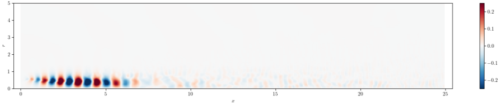


Results

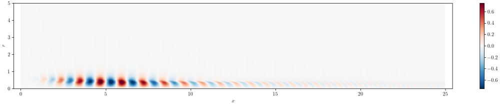
Stochastic model

Mode 0: v, $St = 0.7$

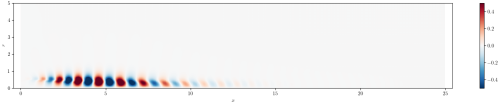
SPOD:



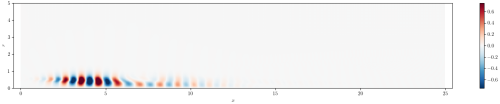
Resolvent:



Resolvent with eddy viscosity:



Stochastic model:

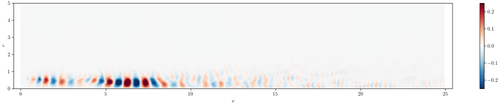


Results

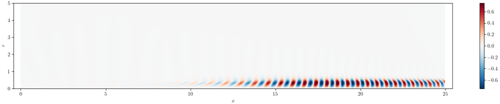
Stochastic model

Mode 1: v, $St = 0.7$

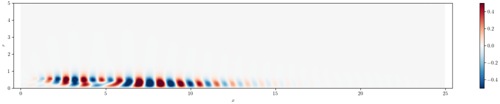
SPOD:



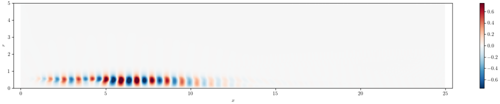
Resolvent:



Resolvent with eddy viscosity:



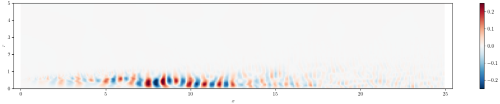
Stochastic model:



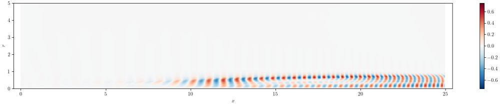
Results

Mode 4: v, $St = 0.7$

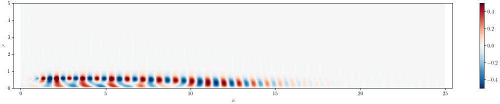
SPOD:



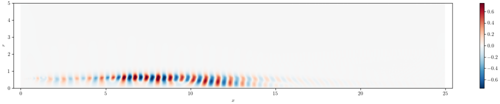
Resolvent:



Resolvent with eddy viscosity:



Stochastic model:

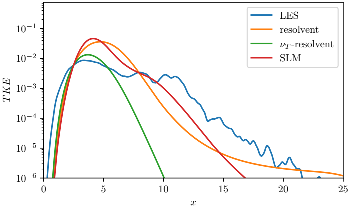


Results

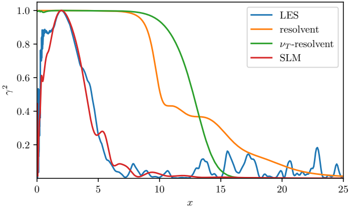
Stochastic model

$St = 0.7$

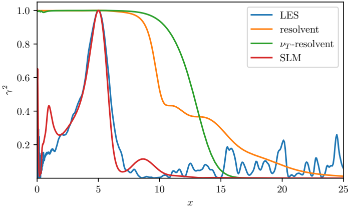
TKE:



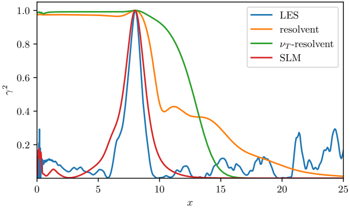
Coherence, v, $x = 2$:



Coherence, v, $x = 5$:



Coherence, v, $x = 8$:

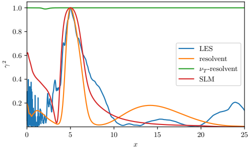


Results

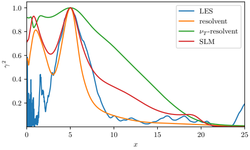
Stochastic model

Coherence, v, $x = 5$

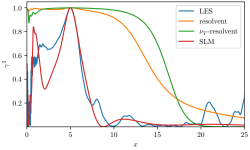
$St = 0.1$:



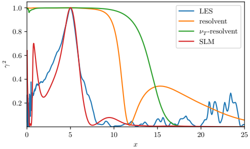
$St = 0.2$:



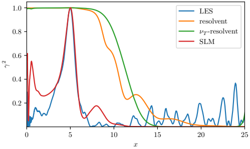
$St = 0.4$:



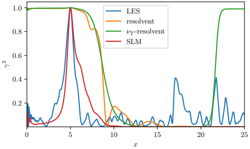
$St = 0.6$:



$St = 0.8$:



$St = 1.0$:



Results

Acoustic predictions

Reba et al. (2010)
Baqui et al. (2015)

Acoustic propagation:

- Pressure CSD at a Kirchhoff surface (cylindrical $r = 1.3D$), Near field incompressible solution.
- Coherence matching on 1-mode wavepacket envelop.
- Fourier transform in x.
- Green's functions (Hankel functions) propagation $r = 10D$.
- Inverse Fourier transform $\mapsto$ CSD, then SPL.

Caveats: Limited validity range ($M = 0.4$, domain size).

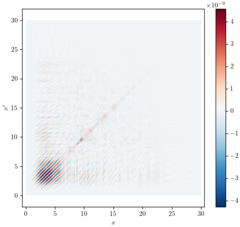
Results

Acoustic predictions

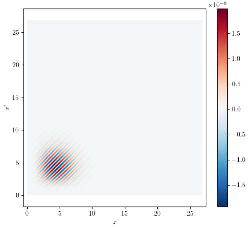
$St = 0.7$

Pressure Kirchhoff surface: $\mathrm{real}(\boldsymbol{S}^{pp}_{m,\omega}(x, r = 1.3, x', r = 1.3))$

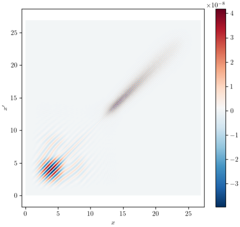
SPOD:



Resolvent:



Stochastic:



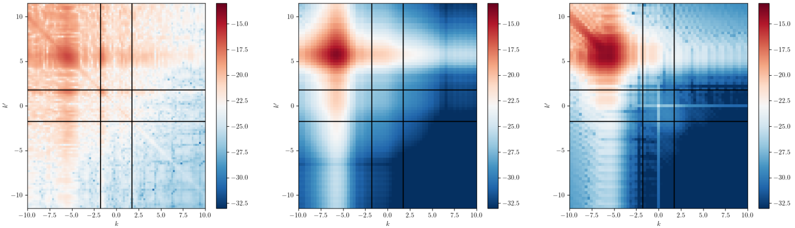
Results

Acoustic predictions

$St = 0.7$

Pressure Kirchhoff surface: $|\hat{\boldsymbol{S}}^{pp}_{m,\omega}(k, r = 1.3, k', r = 1.3)|^2$

SPOD: Resolvent: Stochastic:



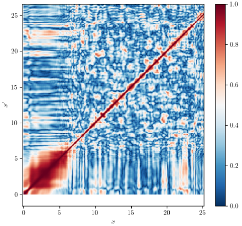
Results

Acoustic predictions

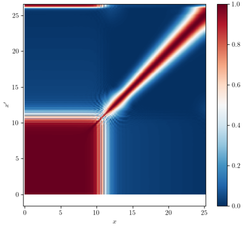
$St = 0.7$

Pressure Kirchhoff surface: $\gamma^{pp}_{m,\omega}(x, r = 1.3, x', r = 1.3)$

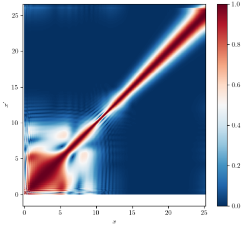
SPOD:



Resolvent:



Stochastic:

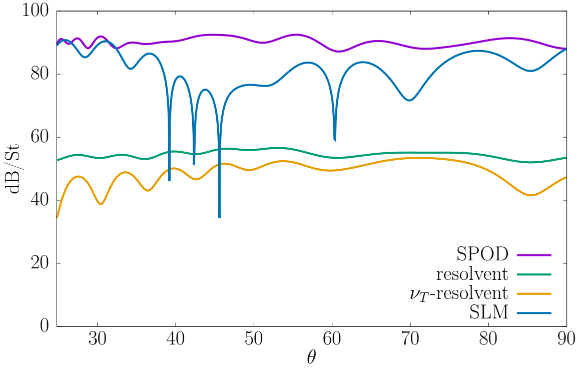


Results

Acoustic predictions

$St = 0.7$

Pressure propagated: at $r = 10D$, $|p|^2_{m,\omega}$



Conclusion

Summary:

- Wavepacket submitted to stochastic transport.
- Stochastic noise, drift velocity, stochastic diffusion.
- Coherence decay prediction.
- Encouraging for subsonic turbulent jets acoustics.
- Perspectives to extend to compressible framework.

Preliminaries in stochastic calculus: Naïve approach

$$\frac{\partial \boldsymbol{q}}{\partial t} = A\boldsymbol{q} + \underbrace{\boldsymbol{f}}_{\text{Stochastic variable}}$$

Preminiaries

Do not divide by $\mathrm{d}t$!

…see just later why.

Preminiaries

Naïve approach: $\mathrm{d}\boldsymbol{q} = \int_t^{t+\mathrm{d}t} \mathrm{d}\boldsymbol{q} = \boldsymbol{q}(t+\mathrm{d}t) - \boldsymbol{q}(t)$

$$\mathrm{d}\boldsymbol{q} = A\boldsymbol{q}\,\mathrm{d}t + \boldsymbol{f}\,\mathrm{d}t$$

Preminiaries

Naïve approach: $\mathrm{d}\boldsymbol{q} = \int_t^{t+\mathrm{d}t} \mathrm{d}\boldsymbol{q} = \boldsymbol{q}(t+\mathrm{d}t) - \boldsymbol{q}(t)$

$$\mathrm{d}\boldsymbol{q} = A\boldsymbol{q}\,\mathrm{d}t + \boldsymbol{f}\,\mathrm{d}t$$

$$\mathrm{d}E_f = \mathbb{E}\left(\int_t^{t+\mathrm{d}t} \boldsymbol{q}\cdot\boldsymbol{f}\,\mathrm{d}s\right)$$

$$= \mathbb{E}\left(\int_t^{t+\mathrm{d}t}\left(\int_t^s \boldsymbol{f}\,\mathrm{d}s'\right)\cdot\boldsymbol{f}\,\mathrm{d}s\right)$$

$$= \mathbb{E}\left(\frac{1}{2}\|\boldsymbol{f}\|^2\,\mathrm{d}t^2\right)$$

$$\frac{\mathrm{d}E_f}{\mathrm{d}t} = \frac{1}{2}\mathbb{E}\left(\|\boldsymbol{f}\|^2\right)\ \mathrm{d}t$$

Preminiaries

Naïve approach: $\mathrm{d}\boldsymbol{q} = \int_t^{t+\mathrm{d}t} \mathrm{d}\boldsymbol{q} = \boldsymbol{q}(t+\mathrm{d}t) - \boldsymbol{q}(t)$

$$\mathrm{d}\boldsymbol{q} = A\boldsymbol{q}\,\mathrm{d}t + \boldsymbol{f}\,\mathrm{d}t \qquad\qquad \mathrm{d}\boldsymbol{q} = A\boldsymbol{q}\,\mathrm{d}t + \boldsymbol{f}\,\mathrm{d}B_t$$

$$\mathrm{d}E_f = \mathbb{E}\left(\int_t^{t+\mathrm{d}t} \boldsymbol{q}\cdot\boldsymbol{f}\,\mathrm{d}s\right)$$

$$= \mathbb{E}\left(\int_t^{t+\mathrm{d}t}\left(\int_t^{s} \boldsymbol{f}\mathrm{d}s'\right)\cdot\boldsymbol{f}\,\mathrm{d}s\right)$$

$$= \mathbb{E}\left(\frac{1}{2}\|\boldsymbol{f}\|^2\,\mathrm{d}t^2\right)$$

$$\frac{\mathrm{d}E_f}{\mathrm{d}t} = \frac{1}{2}\mathbb{E}\left(\|\boldsymbol{f}\|^2\right)\ \mathrm{d}t$$

Preminiaries

Naïve approach: $\mathrm{d}\boldsymbol{q} = \int_t^{t+\mathrm{d}t} \mathrm{d}\boldsymbol{q} = \boldsymbol{q}(t+\mathrm{d}t) - \boldsymbol{q}(t)$

$$\mathrm{d}\boldsymbol{q} = A\boldsymbol{q}\,\mathrm{d}t + \boldsymbol{f}\,\mathrm{d}t \qquad\qquad \mathrm{d}\boldsymbol{q} = A\boldsymbol{q}\,\mathrm{d}t + \boldsymbol{f}\,\mathrm{d}B_t$$

$$\begin{aligned}\mathrm{d}E_f &= \mathbb{E}\left(\int_t^{t+\mathrm{d}t} \boldsymbol{q}\cdot\boldsymbol{f}\,\mathrm{d}s\right) \\ &= \mathbb{E}\left(\int_t^{t+\mathrm{d}t}\left(\int_t^s \boldsymbol{f}\mathrm{d}s'\right)\cdot\boldsymbol{f}\,\mathrm{d}s\right) \\ &= \mathbb{E}\left(\frac{1}{2}\|\boldsymbol{f}\|^2\,\mathrm{d}t^2\right)\end{aligned}$$

$$\frac{\mathrm{d}E_f}{\mathrm{d}t} = \frac{1}{2}\mathbb{E}\left(\|\boldsymbol{f}\|^2\right)\,\mathrm{d}t$$

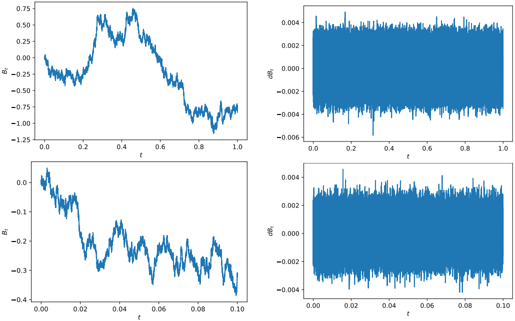
$$\begin{aligned}\mathrm{d}E_f &= \mathbb{E}\left(\int_t^{t+\mathrm{d}t} \boldsymbol{q}\cdot\boldsymbol{f}\,\mathrm{d}B_s\right) \\ &= \mathbb{E}\left(\int_t^{t+\mathrm{d}t}\left(\int_t^s \boldsymbol{f}\,\mathrm{d}B_s'\right)\cdot\boldsymbol{f}\,\mathrm{d}B_s\right) \\ &= \mathbb{E}\left(\frac{1}{2}\|\boldsymbol{f}\|^2\,\mathrm{d}t\right)\end{aligned}$$

$$\frac{\mathrm{d}E_f}{\mathrm{d}t} = \frac{1}{2}\mathbb{E}\left(\|\boldsymbol{f}\|^2\right)$$

Use Brownian motions.

Preminiaries

Brownian motion:



B_t $\mathrm{d}B_t$

- $\cancel{\frac{\mathrm{d}B_t}{\mathrm{d}t}}$
- $\mathbb{E}(B_t) = 0$
- $\langle B_t, B_t \rangle_t = t$
- $\mathrm{d}\langle B_t, B_t \rangle_t = \mathrm{d}t$

Covariation:

$$\langle X, Y \rangle_t = \lim_{\mathrm{d}s \mapsto 0} \sum_{i=0}^{N} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$$

Itō Calculus:

$$X_t = \int_0^t u_X(s)\,\mathrm{d}s + \int_0^t \sigma_X(s)\,\mathrm{d}B_s$$

$$Y_t = \int_0^t u_Y(s)\,\mathrm{d}s + \int_0^t \sigma_Y(s)\,\mathrm{d}B_s$$

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Itō Calculus:

$$\mathrm{d}X_t = u_X \,\mathrm{d}t + \sigma_X \,\mathrm{d}B_t$$
$$\mathrm{d}Y_t = u_Y \,\mathrm{d}t + \sigma_Y \,\mathrm{d}B_t$$

Itō Calculus:

$$\mathrm{d}X_t = u_X \,\mathrm{d}t + \sigma_X \,\mathrm{d}B_t$$
$$\mathrm{d}Y_t = u_Y \,\mathrm{d}t + \sigma_Y \,\mathrm{d}B_t$$

$$\mathrm{d}(XY)_t = X_t \mathrm{d}Y_t + Y_t \mathrm{d}X_t + \mathrm{d}\langle X, Y\rangle_t$$

Itō Calculus:

$$\mathrm{d}X_t = u_X\,\mathrm{d}t + \sigma_X\,\mathrm{d}B_t$$
$$\mathrm{d}Y_t = u_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}B_t$$

$$\mathrm{d}(XY)_t = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t + \mathrm{d}\langle X, Y\rangle_t$$

with

$$\mathrm{d}\langle X, Y\rangle_t = \mathrm{d}\left\langle \int_0^t \sigma_X(s)\mathrm{d}B_s, \int_0^t \sigma_y(s)\mathrm{d}B_s \right\rangle_t = \sigma_X(t)\sigma_y(t)\mathrm{d}t$$

Do not forget covariations.