Coherence decay of wavepackets in turbulent jets by stochastic modelling under location uncertainty

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Motivation: the jet noise problem

- Jet noise dominant during take off.
- Becomes limiting for certifications.
- Noise comes from the flow.

G. Daviller (2011).

São Paulo airport.
Motivation: the jet noise problem

Where does the noise come from?

- Turbulent stochastic eddies?
- Or something more organised?

- Wavepackets in pressure/velocity field.

Tinney & Jordan 2008; Co-axial transonic heated jet $Re = 5 \times 10^6$. Near-field pressure.

- Acoustic directivity as extended source (low azimuthal angles).
Motivation: the jet noise problem

Where does the noise come from?

- Turbulent stochastic eddies?
- Or something more organised?

- Wavepackets in pressure/velocity field.

*Tinney & Jordan 2008; Co-axial transonic heated jet $Re = 5 \times 10^6$.*

Near-field pressure.

- Acoustic directivity as extended source (*low azimuthal angles*).

**Source: likely a wavepacket shape.**
Motivation: the jet noise problem

Wavepackets: Propagated linear instability waves

Experimental subsonic jet mean flow.

Locally-parallel Stability

Jordan & Colonius (2013)
Cavalieri et al. (2013)
Sinha et al. (2014)
Baqui et al. (2015)
Motivation: the jet noise problem

Wavepackets: Propagated linear instability waves

Experimental subsonic jet mean flow.

Centerline $|u|^2$:

Far-field sound:

Non-linearities important for far-field prediction.
Motivation: the jet noise problem

What is missing?

- Propagation condition: \( c = \frac{\omega}{|k|}, \quad k_y = \sqrt{\frac{\omega^2}{c^2} - k_x^2} \)
  \( \Rightarrow \) supersonic axial phase velocity.

- Wavepacket envelop shape allows propagation (due to convolution), but not enough.

- Intensification by turbulence.

From Jordan & Colonius (2013)
Motivation: the jet noise problem

Several representations for intensifying phenomena

- Non-linear term.
- Jittering wavepacket.
- Coherence (2-points statistics).

Cavalieri et al. (2011, 2013)
Baqui et al. (2015)
Motivation: the jet noise problem

Several representations for intensifying phenomena

- Non-linear term.
- Jittering wavepacket.
- Coherence (2-points statistics).

Coherence-matched wavepackets: right acoustic emission

From Baqui et al. (2015)

Cavalieri et al. (2011, 2013)
Baqui et al. (2015)
Motivation: the jet noise problem

Objective:
Simplified model for wavepackets (or coherent structures) evolving within turbulent flows.

- Model-based.
- Predict coherence decay.
- Stochastic modelling under location uncertainty.
Outline

1. Motivation: the jet noise problem
2. Classical methods
3. Stochastic model under location uncertainty
4. Results
Classical methods

Extract from data:

- Proper Orthogonal Decomposition (POD).
- Dynamic Mode Decomposition (DMD), and variants...
- Spectral Proper Orthogonal Decomposition (SPOD).

Models:

- Linear stability.
- PSE, OWNS.
- Resolvent analysis.
**Classical methods**

**SPOD**

**Turbulent signal:**

\[ C_{ij}(x, x', \theta, \tau) = \langle u_i(x, \theta_0, t)u_j(x', \theta_0 + \theta, t + \tau) \rangle \quad x = (x, r)^T \]

(From Pope 2000).

(From Jimenez 2013).

**Fourier transform in** \((\theta, t)\)

\[ S^{ij}_{m, \omega}(x, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{2\pi} C_{ij}(x, x', \theta, \tau)e^{i(\omega\tau - 2\pi m \theta)} d\tau d\theta, \]
Classical methods

Some quantities:

- Cross power spectral density tensor (CSD): $S_{m,\omega}^{ij}(x,x')$.
- Power spectral density (PSD): $S_{m,\omega}^{ii}(x,x)$.
- Coherence: $\gamma_{m,\omega,i,j}^{2}(x,x') = \frac{|S_{m,\omega}^{ij}(x,x')|^2}{S_{m,\omega}^{ii}(x,x)S_{m,\omega}^{jj}(x',x')}$
  
  $\gamma = 1$: Perfectly synchronised.
  $\gamma = 0$: Decorrelated.

Mixing layer $Re = 300$, Spanwise vorticity $\xi$. 
Classical methods

SPOD

Spectral proper orthogonal decomposition (SPOD):

\[ S_{m,\omega} W \Phi_{k,m,\omega}^{SPOD} = \lambda_{k,m,\omega} \Phi_{k,m,\omega}^{SPOD}, \]

Interest:

- Perfectly coherent modes.
- Decorrelated from each other.
- Orthonormal bases.
- Sorted by energy content.

Towne et al. (2018)
Spectral proper orthogonal decomposition (SPOD):

\[ S_{m,\omega} W \Phi_{k,m,\omega}^{\text{SPOD}} = \lambda_{k,m,\omega} \Phi_{k,m,\omega}^{\text{SPOD}}, \]

Interest:

- Perfectly coherent modes.
- Decorrelated from each other.
- Orthonormal bases.
- Sorted by energy content.

Low order representation:

\[ S_{m,\omega}(x, x') \approx \sum_{k=1}^{N} \Phi_{k,m,\omega}^{\text{SPOD}}(x) \lambda_{k,m,\omega} \Phi_{k,m,\omega}^{*,\text{SPOD}}(x') \]

And then access to \( \gamma^2 \).
Classical methods

Navier-Stokes equations:

\[
\begin{cases}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} \\
\nabla \cdot \mathbf{u} = 0.
\end{cases}
\]
Classical methods

Navier-Stokes equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= -\nabla p + \frac{1}{Re} \Delta u \\
\nabla \cdot u &= 0.
\end{align*}
\]

Linearisation: Around the mean flow \( \bar{u} = (U(y), 0, 0)^T \).

\[
\begin{align*}
\frac{\partial u'}{\partial t} + (\bar{u} \cdot \nabla)u' + (u' \cdot \nabla)\bar{u} + \nabla p' - \frac{1}{Re} \Delta u' &= -(u' \cdot \nabla)u' \\
\nabla \cdot u' &= 0.
\end{align*}
\]
Classical methods

Navier-Stokes equations:

\[
\begin{cases}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} \\
\nabla \cdot \mathbf{u} = 0.
\end{cases}
\]

Linearisation: Around the mean flow \( \bar{\mathbf{u}} = (U(y), 0, 0)^T \).

\[
\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'
\]

\[
\begin{cases}
\frac{\partial \mathbf{u}'}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \bar{\mathbf{u}} + \nabla p' - \frac{1}{Re} \Delta \mathbf{u}' = -(\mathbf{u}' \cdot \nabla) \mathbf{u}' \\
\nabla \cdot \mathbf{u}' = 0.
\end{cases}
\]

Fourier transform:

\[
\mathbf{u}'(x, r, \theta, t) = \frac{1}{2\pi} \sum_m \int_{-\infty}^{\infty} \hat{\mathbf{u}}_{m, \omega}(x, r) e^{i(-\omega t + 2\pi m \theta)} \, d\omega.
\]
Non-linearity as an “external forcing”

Navier-Stokes in the frequency-wavenumber domain:

\[
(A\bar{u}, m, \omega - i\omega E) \hat{q}_{m, \omega} = \mathcal{N}_{\bar{u}, m, \omega}(q') = \mathcal{F}(-u' \cdot \nabla u').
\]
Classical methods

Non-linearity as an “external forcing”

Navier-Stokes in the frequency-wavenumber domain:

\[
(A\bar{u},m,\omega - i\omega E) \hat{q}_{m,\omega} = \mathcal{N}_{\bar{u},m,\omega}(q') = \mathcal{F}(-u' \cdot \nabla u').
\]

Identify relevant non-linearities.
Non-linearity as an “external forcing”

Navier-Stokes in the frequency-wavenumber domain:

\[(A\bar{u},m,\omega - i\omega E)\hat{q}_{m,\omega} = N\bar{u},m,\omega(q') = \mathcal{F}(-u' \cdot \nabla u').\]
Classical methods

Linearised problem with external forcing:

\[ H \hat{q}_{m,\omega} = H(A\tilde{u},m,\omega - i\omega E)^{-1} B \hat{f}_{m,\omega} \]

\[ \text{SVD} \]
Classical methods

Linearised problem with external forcing:

\[ H\hat{q}_{m,\omega} = H(A\bar{u},m,\omega - i\omega E)^{-1}B\hat{f}_{m,\omega}. \]

SVD to maximise Rayleigh quotient:

\[
\max_{\hat{f}_{m,\omega}} \frac{\|H\hat{q}_{m,\omega}\|^2}{\|\hat{f}_{m,\omega}\|^2} = \frac{\|\left(H(A\bar{u},m,\omega - i\omega E)^{-1}B\right)\hat{f}_{m,\omega}\|^2}{\|\hat{f}_{m,\omega}\|^2}.
\]
Classical methods

Linearised problem with external forcing:

\[ H\hat{q}_{m,\omega} = H(A\bar{u},m,\omega - i\omega E)^{-1}B\hat{f}_{m,\omega}. \]

SVD to maximise Rayleigh quotient:

\[
\max_{\hat{f}_{m,\omega}} \frac{\|H\hat{q}_{m,\omega}\|^2}{\|\hat{f}_{m,\omega}\|^2} = \frac{\left\|\left(H(A\bar{u},m,\omega - i\omega E)^{-1}B\right)\hat{f}_{m,\omega}\right\|^2}{\|\hat{f}_{m,\omega}\|^2}.
\]

Most amplified harmonic forcing/response modes:

\[ H(A\bar{u},m,\omega - i\omega E)^{-1}B = U\Sigma V^* \]

\[ H(A\bar{u},m,\omega - i\omega E)^{-1}BV_i = \sigma_i U_i \]

with \( U = (U_1, \ldots, U_N), V = (V_1, \ldots, V_N) \) and \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_N) \).
Classical methods

Link with SPOD:

If \((\mathbf{u} \cdot \nabla)\mathbf{u}\) (i.e. \(\mathbf{f}\) in the data) is a Gaussian white noise, then SPOD and resolvent modes match.

Proof:

\[
\mathcal{R} = H (A \bar{\mathbf{u}}, m, \omega - i \omega E)^{-1} B
\]

\[
\mathbf{S}_{m, \omega} = \mathbb{E} (\hat{q}_{m, \omega} \hat{q}_{m, \omega}^*) = \mathcal{R} \mathbb{E} \left( \hat{f}_{m, \omega} \hat{f}_{m, \omega}^* \right) \mathcal{R}^* \]

Then, eigenfunctions of \(\mathbf{S}_{m, \omega}\) (SPOD) are resolvent modes.
Classical methods

Eddy viscosity

Triple decomposition

\[ u(x, t) = \bar{u}(x) + \tilde{u}(x, t) + u'(x, t) \]

- average
- coherent
- turbulent

Ensemble/phase averaging \( \langle u \rangle = \bar{u} + \tilde{u} \).
Classical methods

Eddy viscosity

Triple decomposition

\[ u(x, t) = \bar{u}(x) + \tilde{u}(x, t) + u'(x, t) \]

average \hspace{1cm} coherent \hspace{1cm} turbulent

Ensemble/phase averaging \( \langle u \rangle = \bar{u} + \tilde{u} \).

We obtain generalised Reynolds stresses

\[ \mathcal{F} \left( (\bar{u} \cdot \nabla) \tilde{u} - (\bar{u} \cdot \nabla) \tilde{u} \right) + \mathcal{F} \left( (u' \cdot \nabla) u' - (u' \cdot \nabla) u' \right) \]

\[ \approx \nabla \cdot (\nu_t \nabla \tilde{u}) \]
**Classical methods**

**Eddy viscosity**

**Triple decomposition**

\[ u(x, t) = \bar{u}(x) + \tilde{u}(x, t) + u'(x, t) \]

\[ \text{average} \quad \text{coherent} \quad \text{turbulent} \]

Ensemble/phase averaging \( \langle u \rangle = \bar{u} + \tilde{u} \).

We obtain generalised Reynolds stresses

\[
F \left( \left( \bar{u} \cdot \nabla \right) \bar{u} - \left( \bar{u} \cdot \nabla \right) \tilde{u} \right) + F \left( \left( u' \cdot \nabla \right) u' - \left( u' \cdot \nabla \right) u' \right) \\
\approx \nabla \cdot (\nu_t \nabla \tilde{u})
\]

- Models energy transfers from scales with high production.
- \( \nu_t(\omega = 0) \neq \nu_t(\omega \neq 0) \).
- Purely dissipative \( \Rightarrow \) forbid backscatter.

Reynolds & Hussain (1972)
Morra et al. (2019)
Symon et al. (2020)
Motivation

Classical methods

Stochastic model

Results

Conclusion

Main idea

Modelling under location uncertainty

Stochastic model

Decomposition: \( \mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}) + \tilde{\mathbf{u}}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t) \)

average

coherent

turbulent

G. Tissot, A. Cavalieri, & E. Mémin

Input-output analysis of the stochastic Navier Stokes equations: Application to turbulent channel flow

Phys. Rev. Fluids (2023)

G. Tissot, A. Cavalieri, & E. Mémin

Stochastic linear modes in a turbulent channel flow,

J. Fluid Mech. (2021)
Stochastic model

Particle displacement

\[ X(x, t) = X(x, t_0) + \int_{t_0}^{t} u(x, t)dt + \int_{t_0}^{t} \sigma dB_t , \]

Brownian motion

Differential form

\[ dX(x, t) = u(x, t)dt + \sigma dB_t \]

Resolved, smooth Unresolved, random

with

\[ \sigma dB_t = \int_{\Omega} \sigma(x, x', t)dB_t(x') dx'. \]
Stochastic model

Stochastic transport operator

\[ d_t \theta + \nabla \cdot (\theta u^*) dt + \nabla \cdot (\theta \sigma dB_t) = \frac{1}{2} \nabla \cdot (a \nabla \theta) dt, \]

with the drift velocity

\[ u^* = u - \frac{1}{2} \nabla \cdot a + \sigma (\nabla \cdot \sigma). \]

and \( a_{ij}(x, t) = \int_{\Omega} \sigma^{ik}(x, y, t) \sigma^{kj}(x, y, t) dy = \sigma \sigma^T. \)
Stochastic Navier-Stokes equations

\[
\begin{aligned}
\frac{du}{dt} + (u^* \cdot \nabla) u \, dt + (\sigma dB_t \cdot \nabla) u &= -\nabla (p_t \, dt + dp_t) \\
&\quad + \frac{1}{Re} \nabla \cdot (\nabla u) \, dt + \frac{1}{2} \nabla \cdot (\alpha \nabla u) \, dt + \frac{1}{Re} \nabla \cdot (\nabla \sigma dB_t) \\
\nabla \cdot u^* &= 0; \quad \nabla \cdot \sigma = 0, \\
u^*_t &= u - \frac{1}{2} \nabla \cdot \alpha.
\end{aligned}
\]

Applications:
- Large-eddy simulations.
- Geophysical flow simulations.

Kunita (1997)  
Mémin (2014)  
Resseguier et al. (2017)  
Chandramouli et al. (2018)  
Li et al. (2023)
Stochastic model

Linearisation $u = \bar{u} + \tilde{u}$; $u' \approx \sigma dB_t$

\[
\begin{aligned}
\frac{d}{dt}\tilde{u} + (\bar{u}^* \cdot \nabla)\tilde{u} dt + (\tilde{u} \cdot \nabla)\bar{u} dt + (\sigma dB_t \cdot \nabla) \bar{u} + (\sigma dB_t \cdot \nabla) \tilde{u} \\
= -\nabla (\tilde{p}_t dt + dp_t) \\
+ \frac{1}{Re} \nabla \cdot (\nabla \tilde{u}) dt + \frac{1}{2} \nabla \cdot (a \nabla u) dt + \frac{1}{Re} \nabla \cdot (\nabla \sigma dB_t) \\
\nabla \cdot \tilde{u} = 0; \quad \nabla \cdot \sigma = 0
\end{aligned}
\]
Stochastic model

Linearisation \( u = \bar{u} + \tilde{u} ; u' \approx \sigma dB_t \)

\[
\begin{align*}
\begin{cases}
 dt \tilde{u} + (\bar{u}^* \cdot \nabla) \tilde{u} \, dt + (\tilde{u} \cdot \nabla) \bar{u} \, dt + (\sigma dB_t \cdot \nabla) \bar{u} + (\sigma dB_t \cdot \nabla) \tilde{u} \\
= -\nabla (\tilde{p}_t \, dt + d \rho_t) \\
+ \frac{1}{Re} \nabla \cdot (\nabla \tilde{u}) \, dt + \frac{1}{2} \nabla \cdot (a \nabla u) \, dt + \frac{1}{Re} \nabla \cdot (\nabla \sigma dB_t) \\
\n\n\end{cases}
\end{align*}
\]

\( \nabla \cdot \tilde{u} = 0 ; \quad \nabla \cdot \sigma = 0 \)

Ansatz

\[
\begin{cases}
\tilde{u}(x, r, \theta, t) = \hat{u}_{m, \omega}(x, r) e^{i(2\pi m \theta - \omega t - \sum_j \omega_j' \zeta_j, t)} \\
\sigma dB_t = d \xi_{m, \omega}(x, r) e^{i(2\pi m \theta - \omega t - \sum_j \omega_j' \zeta_j, t)}
\end{cases}
\]

with \( \sigma dB_t = \sum_{m, j} \Phi_{B, m, j}(x, r) e^{i(2\pi m \theta)} d\zeta_j, t \)
Stochastic model

Random part

$$\sum \omega_j \hat{u}_{m,\omega} \text{d}\zeta_{j,t} = \mathcal{F} (\sigma \text{d}B_t \cdot \nabla \tilde{u}) + \nabla q_j \text{d}\zeta_{j,t},$$

Predicts the random phase... but not used here.
Stochastic model

Stochastic linear model

\[
\begin{pmatrix}
A_{\text{adv}}(\cdot) + \frac{\partial U}{\partial x} - D_{xx}(\cdot) & \frac{\partial U}{\partial r} - D_{xr}(\cdot) & 0 \\
\frac{\partial V}{\partial x} - D_{rx}(\cdot) & A_{\text{adv}}(\cdot) + \frac{\partial V}{\partial r} - D_{rr}(\cdot) & 0 \\
0 & 0 & A_{\text{adv}}(\cdot) + \frac{1}{r} V - D_{zz}(\cdot)
\end{pmatrix}
\begin{pmatrix}
\hat{u}_{m,\omega} \\
\hat{v}_{m,\omega} \\
\hat{w}_{m,\omega} \\
\hat{p}_{m,\omega}
\end{pmatrix}
= \begin{pmatrix}
-(\dot{\xi}_{m,\omega})_x \frac{\partial U}{\partial x} - (\dot{\xi}_{m,\omega})_r \frac{\partial U}{\partial r} + \frac{1}{Re} \nabla \cdot (\nabla (\dot{\xi}_{m,\omega})_x) \\
-(\dot{\xi}_{m,\omega})_x \frac{\partial V}{\partial x} - (\dot{\xi}_{m,\omega})_r \frac{\partial V}{\partial r} + \frac{1}{Re} \nabla \cdot (\nabla (\dot{\xi}_{m,\omega})_y) \\
\frac{1}{Re} \nabla \cdot (\nabla (\dot{\xi}_{m,\omega})_z) \\
0
\end{pmatrix},
\]

with \( A_{\text{adv}}(\cdot) = -i\omega + U_d \frac{\partial \cdot}{\partial x} + V_d \frac{\partial \cdot}{\partial r} \),

and \( D(\cdot) = \frac{1}{Re} \nabla \cdot (\nabla (\cdot)) + \frac{1}{2} \nabla \cdot (a \nabla (\cdot)). \)
Stochastic model

Stochastic linear model

In a compact form

\[ (A_{u,m,\omega}^{\text{SLM}} - i\omega E) \hat{q}_{m,\omega} = B_{\hat{u}}^{\text{SLM}} \hat{f}_{m,\omega} \]

Ensemble ⇒ Cross spectral density matrix and eigenfunctions.Comparable to resolvent and SPOD.
Stochastic model

Stochastic linear model

In a compact form

\[
\left( A_{u,m,\omega}^{\text{SLM}} - i\omega E \right) \hat{q}_{m,\omega} = B_{u}^{\text{SLM}} \hat{f}_{m,\omega}
\]

In practice: solved rewritten as a SVD problem.
Stochastic model

Model definition

\[
\dot{\xi}_{m,\omega} = \sum_{k=2}^{N} \sigma^\text{res}_k \Phi^\text{res}_k \eta_k \sim \mathcal{N}(0,1)
\]

Figure: Schematic representation of the SLM procedure.
Results

Mean flow from RANS: \( M = 0.4, \, Re = 450\,000 \)

\[ U: \]

\[ V: \]

\[ \nu_T: \]
Results

Resolvent analysis: $v, St = 0.7$, focus on $m = 0$

Mode 0:

Mode 1:

Mode 4:
Results

Stochastic model

Spectra

$St = 0.3$

$St = 0.6$

$St = 0.9$
Results

Mode 0: $v$, $St = 0.7$

SPOD:

Resolvent:

Resolvent with eddy viscosity:

Stochastic model:
Mode 1: $v, St = 0.7$

SPOD:

Resolvent:

Resolvent with eddy viscosity:

Stochastic model:
Results

Mode 4: $v$, $St = 0.7$

SPOD:

Resolvent:

Resolvent with eddy viscosity:

Stochastic model:
Results

Stochastic model

$St = 0.7$

TKE:

Coherence, $v$, $x = 2$:

Coherence, $v$, $x = 5$:

Coherence, $v$, $x = 8$:
Results

Stochastic model

Coherence, $v$, $x = 5$

$St = 0.1$: 

$St = 0.2$: 

$St = 0.4$: 

$St = 0.6$: 

$St = 0.8$: 

$St = 1.0$: 

Motivation

Classical methods

Stochastic model

Results

Conclusion
Results

Acoustic predictions

Acoustic propagation:

- Pressure CSD at a Kirchhoff surface (cylindrical $r = 1.3D$), Near field incompressible solution.
- Coherence matching on 1-mode wavepacket envelop.
- Fourier transform in $x$.
- Green’s functions (Hankel functions) propagation $r = 10D$.
- Inverse Fourier transform $\rightarrow$ CSD, then SPL.

Caveats: Limited validity range ($M = 0.4$, domain size).

Reba et al. (2010)
Baqui et al. (2015)
Pressure Kirchhoff surface: \( \text{real}(S_{m,\omega}^{pp}(x, r = 1.3, x', r = 1.3)) \)

\[ S_{m,\omega}^{pp}(x, r = 1.3, x', r = 1.3) \]
Pressure Kirchhoff surface: $|\hat{S}_{m,\omega}^{pp}(k, r = 1.3, k', r = 1.3)|^2$

\begin{figure}
\centering
\begin{tabular}{ccc}
\textbf{SPOD:} & \textbf{Resolvent:} & \textbf{Stochastic:} \\
\includegraphics[width=0.3\textwidth]{spod.png} & \includegraphics[width=0.3\textwidth]{resolvent.png} & \includegraphics[width=0.3\textwidth]{stochastic.png}
\end{tabular}
\end{figure}

\textit{St} = 0.7
Results

Acoustic predictions

Pressure Kirchhoff surface: \( \gamma_{m, \omega}^{pp}(x, r = 1.3, x', r = 1.3) \)

\[ St = 0.7 \]
Results

Acoustic predictions

Pressure propagated: at $r = 10D$, $|p|_{m,\omega}^2$

$St = 0.7$
Conclusion

Summary:

- Wavepacket submitted to stochastic transport.
- Stochastic noise, drift velocity, stochastic diffusion.
- Coherence decay prediction.
- Encouraging for subsonic turbulent jets acoustics.
- Perspectives to extend to compressible framework.
Preliminaries in stochastic calculus: Naïve approach

\[ \frac{\partial q}{\partial t} = Aq + f \]

Stochastic variable
Do not divide by $dt$!

...see just later why.
Naïve approach: \( dq = \int_{t}^{t+dt} dq = q(t + dt) - q(t) \)
\[
dq = Aq \, dt + f \, dt
\]
Naïve approach: \( dq = \int_t^{t+dt} dq = q(t + dt) - q(t) \)
\[
dq = Aq \, dt + f \, dt
\]
\[
dE_f = \mathbb{E} \left( \int_t^{t+dt} q \cdot f \, ds \right)
\]
\[
= \mathbb{E} \left( \int_t^{t+dt} \left( \int_t^s f \, ds' \right) \cdot f \, ds \right)
\]
\[
= \mathbb{E} \left( \frac{1}{2} \| f \|^2 \, dt^2 \right)
\]
\[
\frac{dE_f}{dt} = \frac{1}{2} \mathbb{E} \left( \| f \|^2 \right) \, dt
\]
Naïve approach: \( dq = \int_t^{t+dt} dq = q(t + dt) - q(t) \)
\[
dq = Aq \, dt + f \, dt
\]
\[
dq = Aq \, dt + f \, dB_t
\]
\[
dE_f = \mathbb{E} \left( \int_t^{t+dt} q \cdot f \, ds \right)
\]
\[
= \mathbb{E} \left( \int_t^{t+dt} \left( \int_t^s f \, ds' \right) \cdot f \, ds \right)
\]
\[
= \mathbb{E} \left( \frac{1}{2} \| f \|^2 \, dt^2 \right)
\]
\[
\frac{dE_f}{dt} = \frac{1}{2} \mathbb{E} \left( \| f \|^2 \right) \, dt
\]
Naïve approach: \[ dq = \int_t^{t+dt} dq = q(t + dt) - q(t) \]
\[ dq = Aq \, dt + f \, dt \]

\[ dE_f = \mathbb{E} \left( \int_t^{t+dt} q \cdot f \, ds \right) = \mathbb{E} \left( \int_t^{t+dt} \left( \int_t^s f \, ds' \right) \cdot f \, ds \right) = \mathbb{E} \left( \int_t^{t+dt} \left( \int_t^s f \, dB_s' \right) \cdot f \, dB_s \right) = \mathbb{E} \left( \frac{1}{2} \| f \|^2 \, dt \right) \]

\[ \frac{dE_f}{dt} = \frac{1}{2} \mathbb{E} \left( \| f \|^2 \right) \, dt \]

Use Brownian motions.
Brownian motion:

Covariation:

\[
\langle X, Y \rangle_t = \lim_{d s \to 0} \sum_{i=0}^{N} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})
\]
Itô Calculus:

\[ X_t = \int_0^t u_X(s) \, ds + \int_0^t \sigma_X(s) \, dB_s \]

\[ Y_t = \int_0^t u_Y(s) \, ds + \int_0^t \sigma_Y(s) \, dB_s \]
Itô Calculus:

\[ dX_t = u_X \, dt + \sigma_X \, dB_t \]
\[ dY_t = u_Y \, dt + \sigma_Y \, dB_t \]
Itô Calculus:

\[ dX_t = u_X \, dt + \sigma_X \, dB_t \]
\[ dY_t = u_Y \, dt + \sigma_Y \, dB_t \]
\[ d(XY)_t = X_t \, dY_t + Y_t \, dX_t + d\langle X, Y \rangle_t \]
Itô Calculus:

\[ dX_t = u_X \, dt + \sigma_X \, dB_t \]
\[ dY_t = u_Y \, dt + \sigma_Y \, dB_t \]

\[ d(XY)_t = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t \]

with

\[ d\langle X, Y \rangle_t = d \left\langle \int_0^t \sigma_X(s) dB_s, \int_0^t \sigma_Y(s) dB_s \right\rangle_t = \sigma_X(t) \sigma_Y(t) dt \]

Do not forget covariations.