On Approximation of an Optimal Control Problem for a Linear Elliptic Equation with Unbounded Coefficients

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Control Object: Linear Diffusion Elliptic Equation

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ ($N \geq 3$) with Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N have positive $(N-1)$-dimensional measures.

$$-\mathrm{div}\big(\nabla y + A(x)\nabla y\big) = f \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \Gamma_D,$$
$$\frac{\partial y}{\partial \nu_A} = u \quad \text{on } \Gamma_N,$$

here $u \in L^2(\Gamma_N)$ is a control, $f \in H^{-1}(\Omega)$ is a given distribution.

Matrix of Unbounded Coefficients

In applications, the stream matrix $A(x) = [a_{ij}(x)]_{i,j=1,\dots,N}$ is skew-symmetric, $a_{ij}(x) = -a_{ji}(x)$, measurable and may admit unremovable singularity, namely, belongs to the space $L^2(\Omega; \mathbb{S}^N) = L^2(\Omega)^{\frac{N(N-1)}{2}}$ (rather than L^∞).

Definition

We say that $y = y(A, f, u)$ is a weak solution to the problem

$$-\mathrm{div}\big(\nabla y + A(x)\nabla y\big) = f \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial y}{\partial \nu_A} = u \quad \text{on } \Gamma_N,$$

if $y \in H_0^1(\Omega; \Gamma_D)$ and

$$\int_\Omega (\nabla \varphi, \nabla y)_{\mathbb{R}^N}\, dx + \underbrace{\int_\Omega (\nabla \varphi, A(x)\nabla y)_{\mathbb{R}^N}\, dx}_{[y,\varphi]}$$
$$= \langle f, \varphi \rangle_{H^{-1}(\Omega;\Gamma_D); H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} u\varphi \, d\mathcal{H}^{N-1}, \quad (1)$$

for all elements $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$.

Lemma

Let $u \in L^2(\Gamma_N)$ be a given control. If $y \in H_0^1(\Omega; \Gamma_D)$ is a weak solution to the above boundary value problem, then y belongs to the set D, i.e. the following relation holds true

$$\left| \int_\Omega \left(\nabla \varphi, A(x) \nabla y\right)_{\mathbb{R}^N} dx \right| \leq c(y) \left(\int_\Omega |\nabla \varphi|^2_{\mathbb{R}^N} dx \right)^{1/2} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$$

with some constant $c(y)$ depending on y.

Setting $[y, \varphi] = \int_\Omega \left(\nabla \varphi, A(x) \nabla y\right)_{\mathbb{R}^N} dx, \quad \forall\, y \in D, \ \forall\, \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$, we can define the bilinear form $[y, \varphi]$ for all $\varphi \in H_0^1(\Omega; \Gamma_D)$ using the rule

$$[y, \varphi] = \lim_{\varepsilon \to 0} [y, \varphi_\varepsilon], \tag{2}$$

where $\{\varphi_\varepsilon\}_{\varepsilon > 0} \subset C_0^\infty(\mathbb{R}^N; \Gamma_D)$ and $\varphi_\varepsilon \to \varphi$ strongly in $H_0^1(\Omega; \Gamma_D)$. In this case the value $[v, v]$ is finite for every $v \in D$.

The energy identity

Passing to the limit in

$$\int_\Omega (\nabla \varphi_\varepsilon, \nabla y)_{\mathbb{R}^N}\, dx + \underbrace{\int_\Omega (\nabla \varphi_\varepsilon, A(x)\nabla y)_{\mathbb{R}^N}\, dx}_{[y,\varphi_\varepsilon]}$$

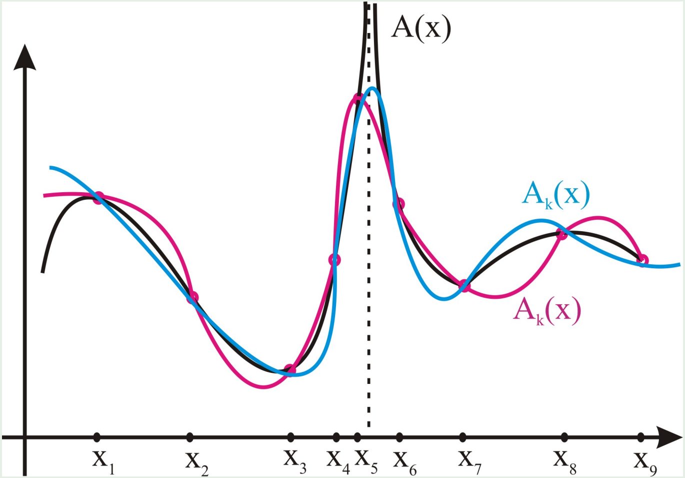
$$= \langle f, \varphi_\varepsilon \rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} u\varphi_\varepsilon\, d\mathcal{H}^{N-1}, \quad (3)$$

as $\varphi_\varepsilon \to y$ in $H_0^1(\Omega;\Gamma_D)$, we immediately come to the relation (the energy identity for weak solutions)

$$\int_\Omega (\nabla y, \nabla y)_{\mathbb{R}^N}\, dx + [y, y] = \langle f, y \rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} uy\, d\mathcal{H}^{N-1}, \quad (4)$$

Example

The main idea of any numerical scheme



Approximable Solutions

A function $y^* = y^*(f,u)$ is called an approximable solution to the above boundary value problem if it can be attained by weak solutions $\{y_k\}_{k\in\mathbb{N}}$ to the boundary value problems

$$-\mathrm{div}\big(\nabla y + A_k(x)\nabla y\big) = f \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \Gamma_D,$$
$$\frac{\partial y}{\partial \nu_{A_k}} = u \quad \text{on } \Gamma_N,$$

where $\{A_k\}_{k=1}^{\infty} \subset L^\infty(\Omega;\mathbb{S}^N)$ are such that $A_k \to A$ strongly in $L^2(\Omega;\mathbb{S}^N)$.

A trivial analysis of numerical approximations. Step 1.

The boundary value problem

$$-\mathrm{div}(\nabla y + A_k(x)\nabla y) = f \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial y}{\partial \nu_{A_k}} = u \quad \text{on } \Gamma_N,$$

has a unique solution $y_k = y(A_k, f, u) \in H_0^1(\Omega; \Gamma_D)$ for every $k \in \mathbb{N}$. Moreover,

$$\int_\Omega (\nabla y_k, \nabla y_k)_{\mathbb{R}^N}\, dx + \underbrace{\int_\Omega (\nabla y_k, A_k(x)\nabla y_k)_{\mathbb{R}^N}\, dx}_{[y_k, y_k]=0}$$
$$= \langle f, y_k\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} u y_k \, d\mathcal{H}^{N-1}, \quad (5)$$

where $A_k \to A$ strongly in $L^2(\Omega; \mathbb{S}^N)$ and $\{A_k\}_{k=1}^\infty \subset L^\infty(\Omega; \mathbb{S}^N)$.

A trivial analysis of numerical approximations. Step 2.

A priory estimate

$$\|y_k\|_{H_0^1(\Omega;\Gamma_D)} \le \|f\|_{H^{-1}(\Omega;\Gamma_D)} + \|u\|_{L^2(\Gamma_N)}, \quad \forall\, k \in \mathbb{N}.$$

Hence,

$$y_k \rightharpoonup y^* \quad \text{weakly in} \quad H_0^1(\Omega;\Gamma_D),$$

where $y^* = y^*(f,u)$ is a weak solution to the original problem

$$-\operatorname{div}\big(\nabla y + A(x)\nabla y\big) = f \quad \text{in } \Omega,$$

$$y = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial y}{\partial \nu_A} = u \quad \text{on } \Gamma_N.$$

A trivial analysis of numerical approximations. Step 3.

Since

$$\int_\Omega (\nabla y_k, \nabla y_k)_{\mathbb{R}^N}\,dx = \langle f, y_k\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} u y_k \, d\mathcal{H}^{N-1}, \quad (6)$$

it follows from the weak convergence $\nabla y_k \rightharpoonup \nabla y^*$ in $L^2(\Omega;\mathbb{R}^N)$ that

$$\int_\Omega (\nabla y^*, \nabla y^*)_{\mathbb{R}^N}\,dx \leq \langle f, y^*\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} u y^* \, d\mathcal{H}^{N-1}. \quad (7)$$

On the other hand, we have the energy identity

$$\int_\Omega (\nabla y^*, \nabla y^*)_{\mathbb{R}^N}\,dx + [y^*, y^*] = \langle f, y^*\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} u y^* \, d\mathcal{H}^{N-1}. \quad (8)$$

Conclusion 1.

If y^* is a weak H^1-limit of approximation solutions $\{y_k\}_{k\in\mathbb{N}}$, then

$$[y^*, y^*] \geq 0 \; !$$

Conclusion 2.

If for a given control $u \in L^2(\Gamma_N)$ there exists an element $v \in D$ such that $[v, v] < 0$ then the boundary value problem

$$-\mathrm{div}\big(\nabla y + A(x)\nabla y\big) = f \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial y}{\partial \nu_A} = u \quad \text{on } \Gamma_N,$$

has more than one weak solution.

The Main Difficulty

The original boundary value problems may admit infinitely many weak solutions which can be divided into two classes: approximable and non-approximable solutions.

Setting of OCP

Let $f \in H^{-1}(\Omega; \Gamma_D)$ and $y_d \in H_0^1(\Omega, \Gamma_D)$ be given distributions and let $A \in L^2(\Omega; \mathbb{S}^N)$ be a given matrix. We consider the following OCP:

$$\text{Minimize } I(u, y) = \|y - y_d\|^2_{H_0^1(\Omega;\Gamma_D)} + \|u\|^2_{L^2(\Gamma_N)} \tag{9}$$

$$\text{subject to constrains}$$

$$-\operatorname{div}\big(\nabla y + A(x)\nabla y\big) = f \quad \text{in} \quad \Omega, \tag{10}$$

$$y = 0 \text{ on } \Gamma_D, \qquad \partial y/\partial \nu_A = u \text{ on } \Gamma_N, \tag{11}$$

$$u \in L^2(\Gamma_N). \tag{12}$$

Here $\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^N \Big(\delta_{ij} + a_{ij}(x)\Big) \frac{\partial y}{\partial x_j} \cos(\nu, x_i)$, δ_{ij} is Kronecker's delta, ν is the outward unit normal vector at Γ_N to Ω.

Definition

We call the pair (u, y) admissible to OCP (9)–(12) if $u \in L^2(\Gamma_N)$, $y \in D$, and (u, y) are related by identity (1). We denote by Ξ the set of all admissible pairs for OCP (9)–(12). We call the pair (u^0, y^0) optimal to (9)–(12) if $(u^0, y^0) \in \Xi$ and $I(u^0, y^0) = \inf_{(u,y)\in\Xi} I(u, y)$.

Theorem

OCP (9)–(12) has a unique solution for each $f \in H^{-1}(\Omega; \Gamma_D)$, $A \in L^2(\Omega; \mathbb{S}^N)$, and $y_d \in H_0^1(\Omega; \Gamma_D)$.

Theorem

If matrix $A \in L^2(\Omega; \mathbb{S}^N)$ is such that $[y, y] = 0 \ \forall\, y \in D$, then a unique solution to (9)–(12) is variational, i.e. it can be attained by solutions to similar OCPs with L^∞-approximated matrix A.

Let $A \in L^2(\Omega; \mathbb{S}^N)$ be a stream matrix. For a given sequence $\{\varepsilon > 0\}$ we consider the cut-off operators $\mathbf{T}_\varepsilon : \mathbb{S}^N \to \mathbb{S}^N$ defined by the rule $\mathbf{T}_\varepsilon(A) = [T_\varepsilon(a_{ij})]_{i,j=1}^N$ $\forall\, \varepsilon > 0$, where $T_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the truncation function

$$T_\varepsilon(s) = \max\left\{\min\left\{s, \varepsilon^{-1}\right\}, -\varepsilon^{-1}\right\}.$$

We associate with such operators the following set of subdomains $\{\Omega_\varepsilon = \Omega \setminus Q_\varepsilon\}_{\varepsilon>0}$ of Ω where

$$Q_\varepsilon = \text{closure}\left\{x \in \Omega \;:\; \|A(x)\|_{\mathbb{S}^N} := \max_{i,j=1,\dots,N} |a_{ij}(x)| \geq \varepsilon^{-1}\right\}.$$

Definition

We call $A \in L^2(\Omega; \mathbb{S}^N)$ a funnel-type matrix if there exists a strictly decreasing sequence $\{\varepsilon\}$ converging to 0 such that the corresponding sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ possess the following properties:

(i) $\Omega_\varepsilon \subset \Omega$ are open connected subsets with Lipschitz boundaries and $\exists\, \delta > 0$, such that $\partial\Omega \subset \partial\Omega_\varepsilon$ and $\operatorname{dist}(\Gamma_\varepsilon, \partial\Omega) > \delta$, $\forall\, \varepsilon > 0$, where $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \partial\Omega$.

(ii) The surface measure of the boundaries of holes $Q_\varepsilon = \Omega \setminus \Omega_\varepsilon$ is small enough: $\mathcal{H}^{N-1}(\Gamma_\varepsilon) = o(\varepsilon) \quad \forall\, \varepsilon > 0$.

(iii) For any element $h \in D$ there is a constant $c(h)$ depending on h and independent of ε such that

$$\left| \int_{Q_\varepsilon} (\nabla\varphi, A(x)\nabla h)_{\mathbb{R}^N}\, dx \right| \le c(h) \sqrt{\frac{|\Omega \setminus \Omega_\varepsilon|}{\varepsilon}} \left(\int_{Q_\varepsilon} |\nabla\varphi|^2_{\mathbb{R}^N}\, dx \right)^{1/2}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$.

Properties of funnel-type matrices

- Each of the sets Ω_ε is locally located on one side of its Lipschitz boundary $\partial\Omega_\varepsilon$;
- $\forall\,\varepsilon\ \partial\Omega_\varepsilon$ can be divided into three parts $\partial\Omega_\varepsilon = \Gamma_D \cup \Gamma_N \cup \Gamma_\varepsilon$;
- the condition $\mathcal{H}^{N-1}(\Gamma_\varepsilon) = o(\varepsilon) \quad \forall\,\varepsilon > 0.$ excludes the appearance of self-similar domains Ω_ε with some fractal behavior of the boundaries;
- the sequence $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is monotonically expanding, i.e., $\Omega_{\varepsilon_k} \subset \Omega_{\varepsilon_{k+1}}$ for all $\varepsilon_k > \varepsilon_{k+1}$, and perimeters of Q_ε tend to zero as $\varepsilon \to 0$, namely $|\Omega \setminus \Omega_\varepsilon| = o(\varepsilon^2)$ and, hence,

$$\frac{\varepsilon\mathcal{H}^{N-1}(\Gamma_\varepsilon)}{|\Omega \setminus \Omega_\varepsilon|} = O(1);$$

- $\chi_{\Omega_\varepsilon} \to \chi_\Omega$ strongly in $L^2(\Omega)$.

Let us consider the following sequence of regularized OCPs associated with domains Ω_ε

$$\left\{ \left\langle \inf_{(u,v,y)\in\Xi_\varepsilon} I_\varepsilon(u,v,y) \right\rangle, \quad \varepsilon \to 0 \right\}, \tag{13}$$

where

$$I_\varepsilon(u,v,y) := \|y - y_d\|^2_{H^1_0(\Omega_\varepsilon;\Gamma_D)} + \|u\|^2_{L^2(\Gamma_N)} + \frac{1}{\varepsilon^\alpha}\|v\|^2_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)}, \tag{14}$$

$$\Xi_\varepsilon = \left\{ (u,v,y) \;\middle|\; \begin{array}{c} -\mathrm{div}\,(\nabla y + A\nabla y) = f \quad \text{in} \quad \Omega_\varepsilon, \\ y = 0 \text{ on } \Gamma_D, \qquad \partial y/\partial \nu_A = u \text{ on } \Gamma_N, \\ \partial y/\partial \nu_A = v \text{ on } \Gamma_\varepsilon, \\ u \in L^2(\Gamma_N),\; v \in H^{-\frac{1}{2}}(\Gamma_\varepsilon),\; y \in H^1_0(\Omega_\varepsilon;\Gamma_D). \end{array} \right\} \tag{15}$$

Here, $y_d \in H^1_0(\Omega,\Gamma_D)$ and $f \in L^2(\Omega)$ are given functions, ν is the outward normal unit vector at Γ_N and Γ_ε to Ω_ε, $v \in H^{-\frac{1}{2}}(\Gamma_\varepsilon)$ is considered as a fictitious control, and α is a positive number such that

$$\varepsilon^{-\alpha}\mathcal{H}^{N-1}(\Gamma_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Theorem

For every $\varepsilon > 0$ there exists a unique minimizer $(u_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ to the problem $\left\langle \inf_{(u,v,y)\in\Xi_\varepsilon} I_\varepsilon(u,v,y)\right\rangle$.

In order to study the asymptotic behavior of the sequences of admissible solutions in the scale of variable spaces, we adopt the following concept.

Definition

We say that a sequence $\{(u_\varepsilon, v_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ weakly converges to $(u, y) \in L^2(\Gamma_N) \times H_0^1(\Omega;\Gamma_D)$ in the scale of spaces

$$\left\{L^2(\Gamma_N) \times H^{-\frac{1}{2}}(\Gamma_\varepsilon) \times H_0^1(\Omega_\varepsilon;\Gamma_D)\right\}_{\varepsilon>0}, \text{ if}$$

$$u_\varepsilon \rightharpoonup u \ \text{ in } \ L^2(\Gamma_N), \quad y_\varepsilon \rightharpoonup y \ \text{ in } \ H_0^1(\Omega_\varepsilon;\Gamma_D),$$

$$\text{and} \quad \sup_{\varepsilon>0} \frac{1}{\mathcal{H}^{N-1}(\Gamma_\varepsilon)} \|v_\varepsilon\|^2_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)} < +\infty.$$

Main Results

Theorem

Let $A \in L^2(\Omega; \mathbb{S}^N)$ be a funnel-type stream matrix such that

the equality $[y, y] = 0$ does not hold in D.

Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a sequence of perforated subdomains of Ω associated with matrix A. Then problem $\langle \inf_{(u,y)\in\Xi} I(u,y) \rangle$, where $y_d \in H_0^1(\Omega, \Gamma_D)$ and $f \in L^2(\Omega)$ are given functions, is a variational limit of sequence (13)–(15) as the parameter ε tends to zero.

Main Results

Theorem

Let $A \in L^2(\Omega; \mathbb{S}^N)$ be a funnel-type stream matrix such that

$$\text{the equality} \quad [y, y] = 0 \quad \text{does not hold in} \quad D.$$

Let $y_d \in H_0^1(\Omega, \Gamma_D)$ and $f \in L^2(\Omega)$ be given functions. Let $\left\{(u_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of optimal solutions to regularized problems (13)–(15). Then a unique optimal solution to OCP (9)–(12) is attainable in the following sense

$$u_\varepsilon^0 \rightharpoonup u^0 \ \text{ in } \ L^2(\Gamma_N), \quad y_\varepsilon^0 \rightharpoonup y^0 \ \text{ in } \ H_0^1(\Omega_\varepsilon; \Gamma_D),$$

$$\inf_{(u,y)\in\Xi} I(u,y) = I\left(u^0, y^0\right) = \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0).$$

Thank you for your unlimited patience

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