Estimation II

Notations

- Observed collection of networks: $\mathcal{A} = \{A^{(m)}, m = 1, \ldots, M\}$
- Latent variables: $\mathcal{U} = (U_1, \ldots, U_M)$, $\mathcal{Z} = \{\mathbf{Z}^{(m)}, m = 1, \ldots, M\}$
- Model parameters: $\theta = \big((p_1, \ldots, p_C), (\boldsymbol{\pi}^{(c)}, \gamma^{(c)})_{c=1,\ldots,C}\big)$

Integrated classification likelihood (ICL)

- Bayesian approach: put a prior $p(\theta)$ on the parameters θ
- Likelihood of complete data (Biernacki et al., 2000):

$$\mathrm{ICL}(\mathcal{U}, \mathcal{Z}; \mathcal{A}) = \log(p(\mathcal{U}, \mathcal{Z}, \mathcal{A})) = \log\left(\int p(\mathcal{U}, \mathcal{Z}, \mathcal{A}|\theta)p(\theta)\mathrm{d}\theta\right)$$

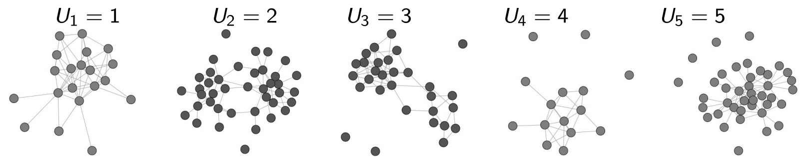
- Estimates of $\mathcal{U}$ and $\mathcal{Z}$ (Côme and Latouche, 2015):

$$(\hat{\mathcal{U}}, \hat{\mathcal{Z}}) = \arg\max_{\mathcal{U},\mathcal{Z}} \mathrm{ICL}(\mathcal{U}, \mathcal{Z}; \mathcal{A})$$

- With an appropriate prior $p(\theta)$, the ICL has closed-form expression
- Discrete optimization

Estimation III

Agglomerative algorithm

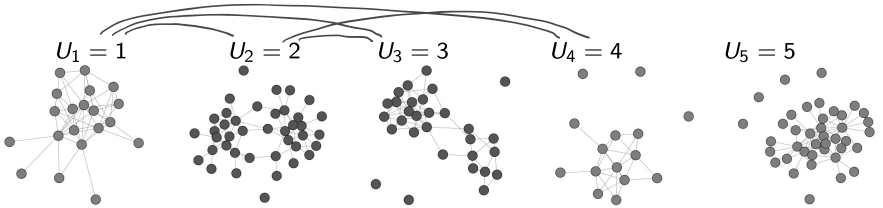


Estimation III

Agglomerative algorithm

1. Compute ICL($\mathcal{U}$) for initial $\mathcal{U}$

2. Compute ICL($\mathcal{U}_{c \cup c'}$) $\forall$ (c,c')



Estimation III

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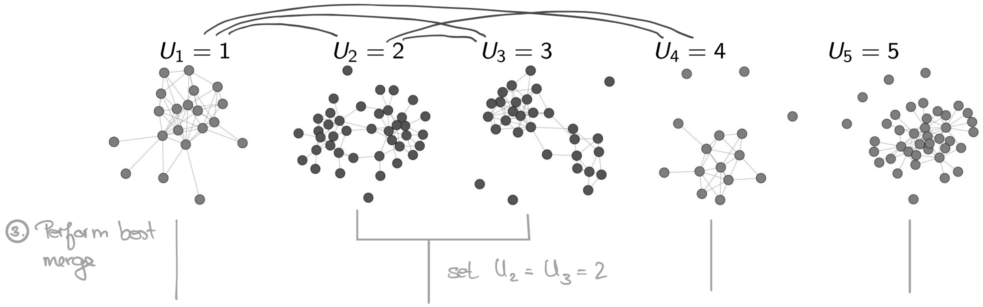

Estimation III

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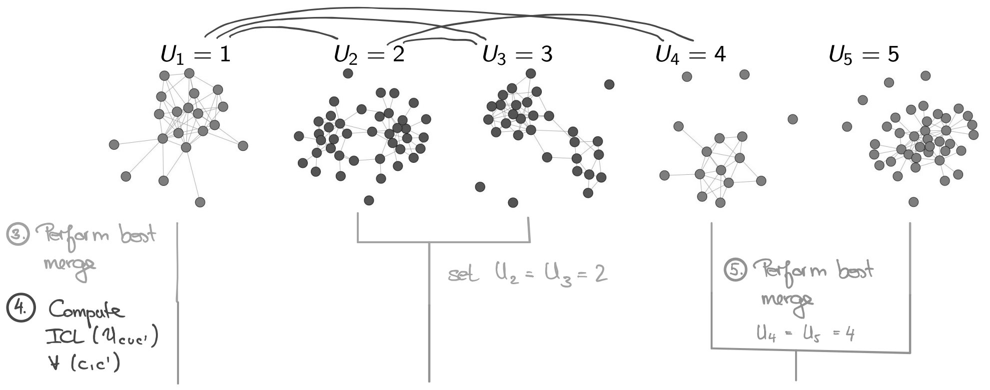

Estimation III

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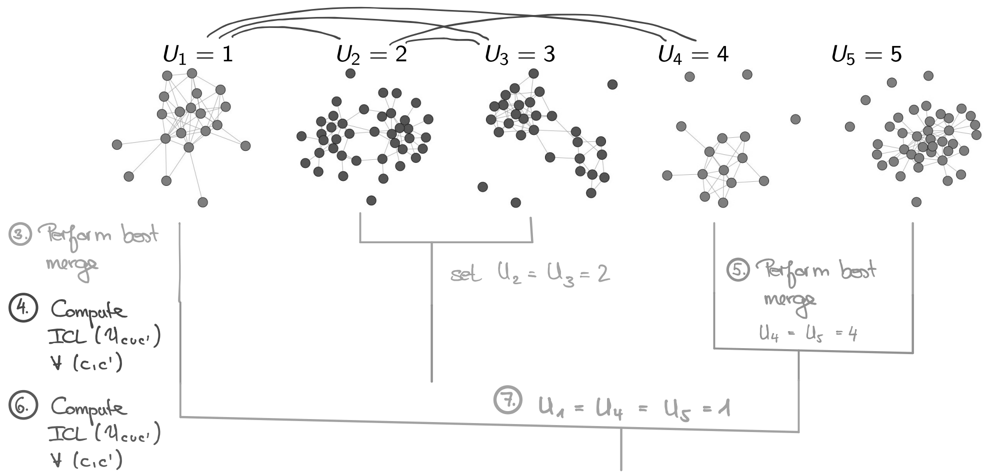
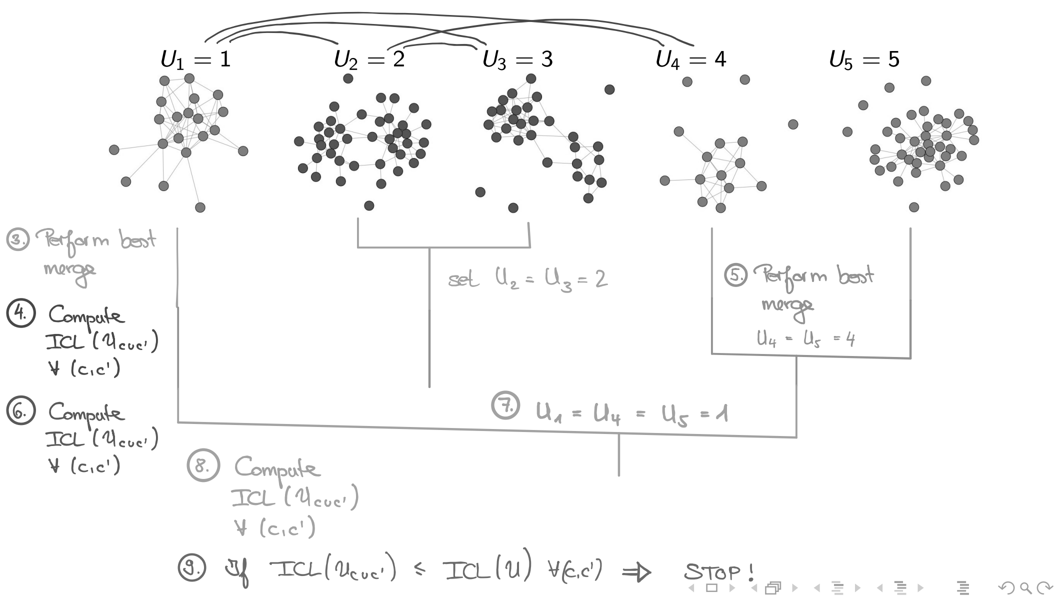


Estimation III

Agglomerative algorithm

(1.) Compute ICL($\mathcal{U}$) for initial $\mathcal{U}$

(2.) Compute ICL($\mathcal{U}_{c \cup c'}$) $\forall$ (c,c')

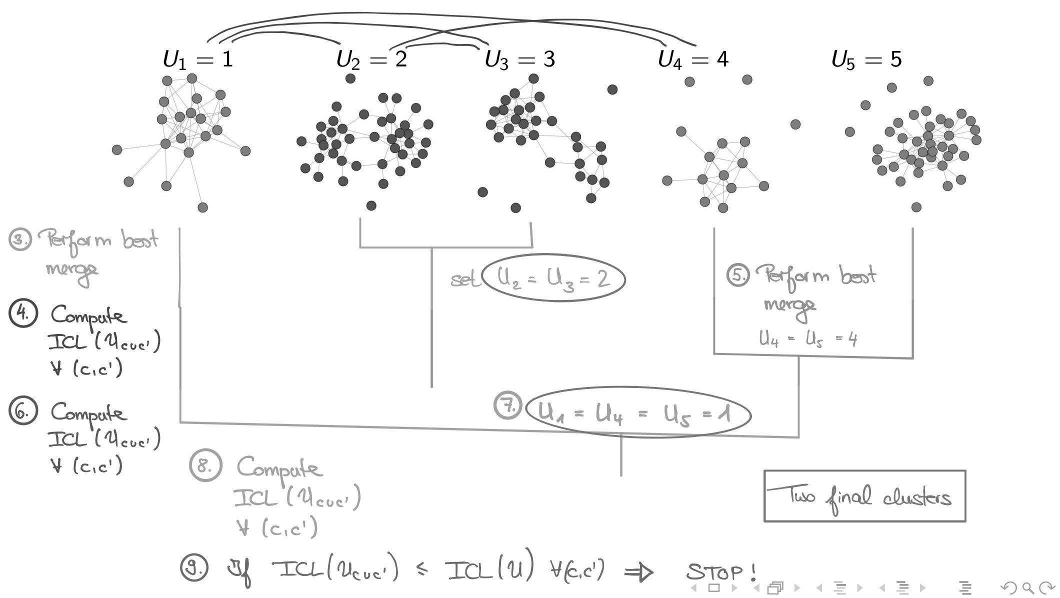


Estimation III

Agglomerative algorithm

(1.) Compute ICL($\mathcal{U}$) for initial $\mathcal{U}$

(2.) Compute ICL($\mathcal{U}_{c \cup c'}$) $\forall$ (c,c')

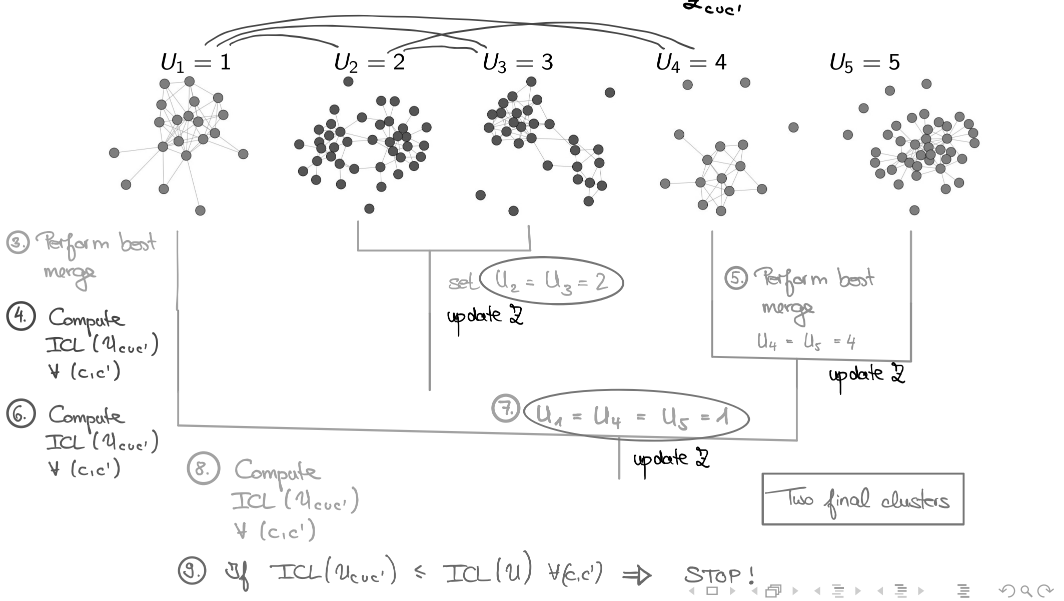


Estimation III

Agglomerative algorithm

1. Compute ICL($\mathcal{U}$) for initial $\mathcal{U}$ (← Z)

2. Compute ICL($\mathcal{U}_{c\cup c'}$) ∀ (c,c') (← $Z_{c\cup c'}$)



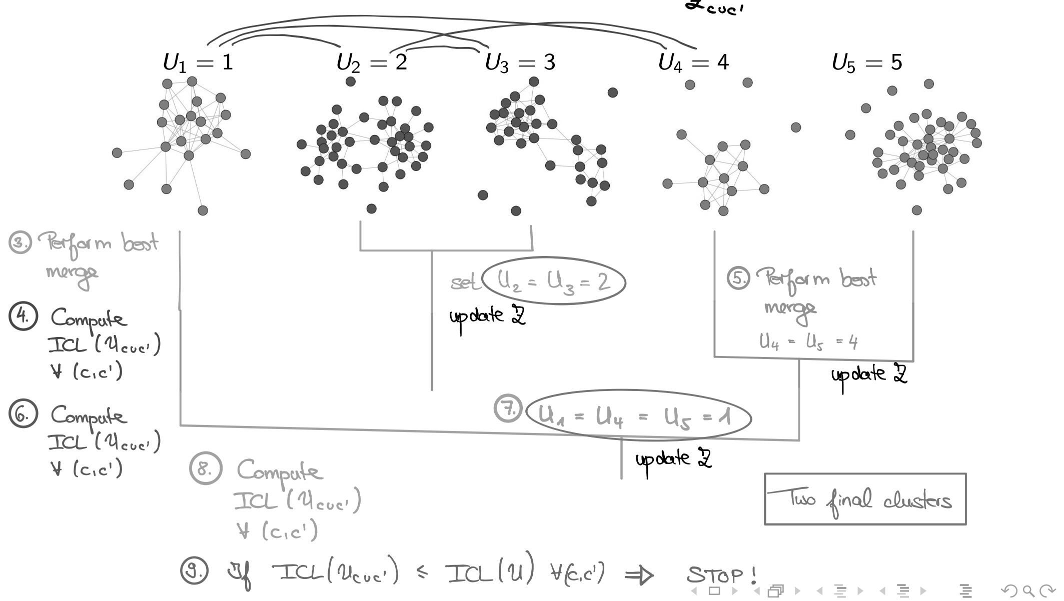
0. Initialization: fit a SBM to each network $A^{(m)}$ → initial $Z^{(m)}$ and $\theta^{(m)}$

Estimation III

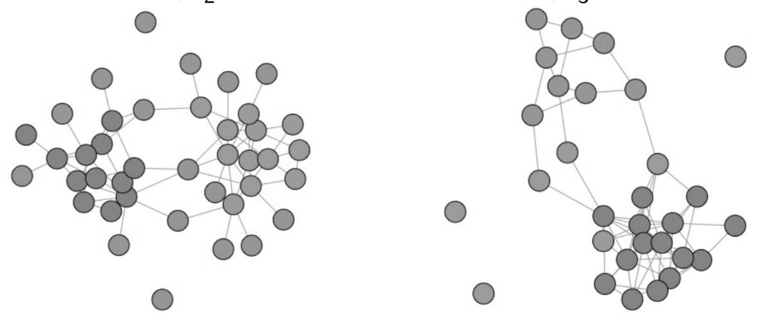
Agglomerative algorithm

1. Compute ICL($\mathcal{U}$) for initial $\mathcal{U}$ (← Z)

2. Compute ICL($\mathcal{U}_{c\cup c'}$) ∀ (c,c') (← $Z_{c\cup c'}$)



Fusion of two clusters I

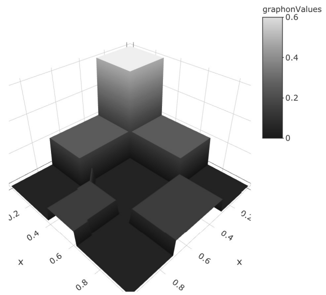


Label-switching problem in the SBM

Fusion of two clusters II

$\boldsymbol{\pi} = (0.3, 0.4, 0.3)$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0 & 0.1 \\ 0 & 0.1 & 0 \end{pmatrix}$$



Graphon of a SBM

- $g(u, v) = \gamma_{k,l}$ for all $u \in I_k$, $v \in I_l$
- intervals $I_k = \left(\sum_{s=1}^{k-1} \pi_s, \sum_{s=1}^{k} \pi_s\right]$
- piecewise constant function
- Graphon g depends on the block labelling!

(Lovász, 2012)

Fusion of two clusters III

- Consider L^2-distance of two graphons:
  $\|g_{\theta_1} - g_{\theta_2}\|_2 = \left(\int_{[0,1]^2}(g_{\theta_1}(u,v) - g_{\theta_2}(u,v))^2 \mathrm{d}(u,v)\right)^{\frac{1}{2}}$

Matching blocks of two SBMs

Search the best permutations of block labels by

$$(\hat{\sigma}_1, \hat{\sigma}_2) \in \arg \min_{\sigma_1 \in \mathfrak{S}_{K_1}, \sigma_2 \in \mathfrak{S}_{K_2}} \|g_{\sigma_1(\theta_1)} - g_{\sigma_2(\theta_2)}\|_2,$$

where $\mathfrak{S}_K$ denotes the set of all permutations of $\{1,\dots,K\}$ and $\sigma(\theta) = \left((\pi_{\sigma(1)},\dots,\pi_{\sigma(K)}),(\gamma_{\sigma(k),\sigma(l)})_{k,l}\right)$.

- does not depend on the clusterings, the data, the number of networks or the number nodes
- works also for $K_1 \neq K_2$