MULTIVARIATE EXTREME VALUE ANALYSIS UNDER A DIRECTIONAL APPROACH

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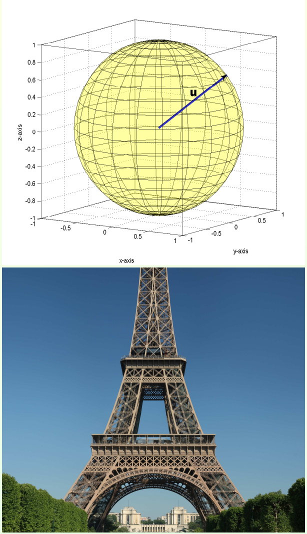
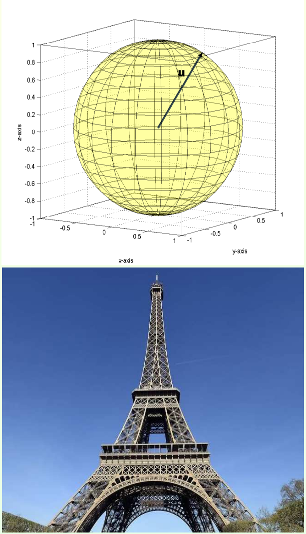
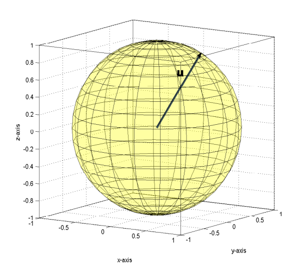
MULTIVARIATE FRAMEWORK AND DIRECTIONS

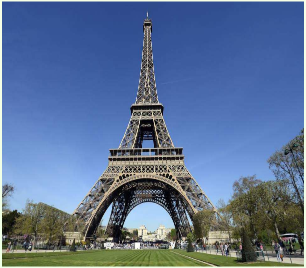


(A) Classical direction $\mathbf{u}$



(B) Auxiliary direction $\mathbf{u}$





Same view, different perspectives

DRAWBACKS IN THE MULTIVARIATE SETTING

- **The lack of a total order in high dimensions.**

Drawbacks in the multivariate setting

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- **The dependence among the variables belonging to a system.**

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Drawbacks in the multivariate setting

- **The lack of a total order in high dimensions.**
- **The dependence among the variables belonging to a system.**
- **There are many interesting directions to analyze the data.**
- **The costs of computing in high dimensions.**

OBJECTIVES

Introduce a directional multivariate setting for extreme value analysis

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1. **Considering the dependence among the variables.**
2. **Giving the possibility of analyzing the variables considering external information, manager preferences or intrinsic system characteristics.**
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OBJECTIVES

Introduce a directional multivariate setting for extreme value analysis

1. **Considering the dependence among the variables.**
2. **Giving the possibility of analyzing the variables considering external information, manager preferences or intrinsic system characteristics.**
3. **Improving the interpretation of the analysis of extremes.**
4. **Providing a non-parametric procedure for estimation to compute the analysis in high dimensions.**

Outline

$$\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} \quad \equiv \quad \textbf{Oriented Orthant.}$$

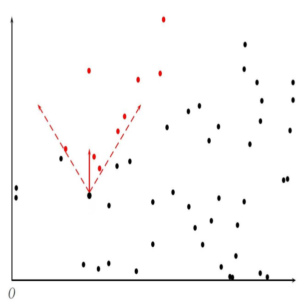
DEFINITION

Given $\mathbf{x}, \quad \mathbf{u} \in \mathbb{R}^n$ and $||\mathbf{u}|| = 1$, the orthant with vertex $\mathbf{x}$ and direction $\mathbf{u}$ is:

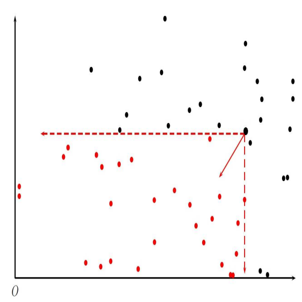
$$\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} = \{\mathbf{z} \in \mathbb{R}^n | R_{\mathbf{u}}(\mathbf{z} - \mathbf{x}) \geq 0\},$$

where $\mathbf{e} = \frac{1}{\sqrt{n}}(1, ..., 1)'$ and $R_{\mathbf{u}}$ is a matrix such that $R_{\mathbf{u}}\mathbf{u} = \mathbf{e}$.

Examples of Oriented Orthants



(A) Orthant in direction $\mathbf{u} = (0, 1)$



(B) Orthant in direction $\mathbf{u} = -\mathbf{e}$

Examples of oriented orthants in $\mathbb{R}^2$

$$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) \quad \equiv \quad \textbf{Directional Multivariate Quantile}$$

DEFINITION

Given $\mathbf{u} \in \mathbb{R}^n$, $||\mathbf{u}|| = 1$ and a random vector $\mathbf{X}$ with distribution probability $\mathbb{P}$, the α-quantile curve in direction $\mathbf{u}$ is defined as:

$$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) := \partial\{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}\left[\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}}\right] \leq \alpha\},$$

where ∂ mans the boundary and $0 \leq \alpha \leq 1$

$$\mathcal{U}_{\mathbf{X}}(\alpha, \mathbf{u}) \equiv \text{Directional Multivariate Upper Level-Set}$$
$$\mathcal{L}_{\mathbf{X}}(\alpha, \mathbf{u}) \equiv \text{Directional Multivariate Lower Level-Set}$$

DEFINITION

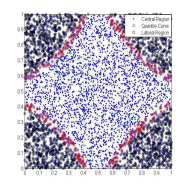
Those sets are defined by:

$$\mathcal{U}_{\mathbf{X}}(\alpha, \mathbf{u}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}[\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}}] < \alpha\},$$

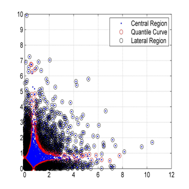
$$\mathcal{L}_{\mathbf{X}}(\alpha, \mathbf{u}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbb{P}[\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}}] > \alpha\}.$$

DIRECTIONAL MULTIVARIATE LEVEL-SETS

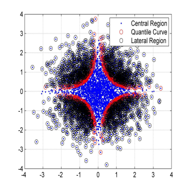
$$\mathbf{u} \in \mathfrak{U} = \left\{ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$$



(A) Bivariate Uniform



(B) Bivariate Exponential

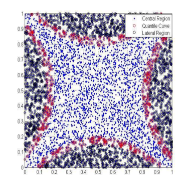


(C) Bivariate Normal

CLASSICAL DIRECTIONS

DIRECTIONAL MULTIVARIATE LEVEL-SETS

$$\mathbf{u} \in \mathfrak{U} = \{(1,0)\,,(0,1)\,,(-1,0)\,,(0,-1)\}$$



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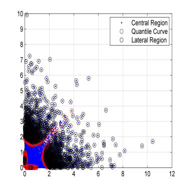

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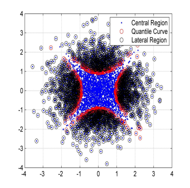


(C) Bivariate Normal

CANONICAL DIRECTIONS

Non-Parametric Estimation

- $\mathbf{X}_m := \{\mathbf{x_1}, \cdots, \mathbf{x_m}\}$, the sample data of the random vector $\mathbf{X}$,
- $\mathbb{P}_{\mathbf{X}_m}[\cdot]$ is the empirical probability law of $\mathbf{X}_m$,
- $\hat{\mathcal{Q}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u}) := \left\{\mathbf{x_j} : |\mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x_j}}\right] - \alpha| \leq h\right\}$ the sample quantile curve with a slack h, avoiding an empty set of estimated quantiles.
- $\hat{\mathcal{U}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u}) := \left\{\mathbf{x_j} : \mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x_j}}\right] < \alpha - h\right\}$ the sample upper $\alpha-$level set with a slack h,
- $\hat{\mathcal{L}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u}) := \left\{\mathbf{x_j} : \mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x_j}}\right] > \alpha + h\right\}$ the sample lowe $\alpha-$level set with a slack h.

NON-PARAMETRIC ESTIMATION

Input: $\mathbf{u}$, α, h and the multivariate sample $\mathbf{X}_m$.

for $i = 1$ to m

$P_i = \mathbb{P}_{\mathbf{X}_m}\left[\mathfrak{C}^{\mathbf{u}}_{\mathbf{x}_i}\right]$,

If $|P_i - \alpha| \leq h$

$\mathbf{x}_i \in \hat{\mathcal{Q}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u})$,

end

If $P_i < \alpha - h$

$\mathbf{x}_i \in \hat{\mathcal{U}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u})$,

end

If $P_i > \alpha + h$

$\mathbf{x}_i \in \hat{\mathcal{L}}^h_{\mathbf{X}_m}(\alpha, \mathbf{u})$,

end

Execution Time

Time in Seconds

Dim\ Size	1000	5000	10000	50000
5	2	49	199	4903
10	2	53	208	5191
50	4	82	325	7656
100	6	139	561	12487

In an Intel core i7 (3,4 GH) computer with 32 Gb RAM.

Extremes through Copulas (Review)

Copula

Roughly speaking, a n-copula C is a particular type of distribution with domain in the unit hyper-cube and uniform margins.

Sklar's Theorem

Let F be a n-dimensional distribution function with marginals $F_1, \cdots, F_n$. Then there exists a n-copula C such that for all $\mathbf{x} \in \bar{\mathbb{R}}^n$,

$$F(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n)).$$

If $F_1, ..., F_n$ are continuous, then C is unique.

$\{\mathbf{v} \in [0,1]^n : C(\mathbf{v}) = \alpha\} \equiv$ **Copula Quantile Procedure**

Let C be the $n-$copula of $\mathbf{X}$ and F_i, $i = 1, \ldots, d$ its margins. Then for $0 < \alpha < 1$ the corresponding $\alpha-$quantile hyper-curve, upper and lower level sets are defined as:

$$\{\mathbf{x} \in \mathbb{R}^n \text{ such that } x_i = F_{X_i}^{-1}(v_i);\ \ i = 1, \ldots, n;\ \ \mathbf{v} \in [0,1]^n \ :\ C(\mathbf{v}) = \alpha\},$$
$$\{\mathbf{x} \in \mathbb{R}^n \text{ such that } x_i = F_{X_i}^{-1}(v_i);\ \ i = 1, \ldots, n;\ \ \mathbf{v} \in [0,1]^n \ :\ C(\mathbf{v}) < \alpha\},$$
$$\{\mathbf{x} \in \mathbb{R}^n \text{ such that } x_i = F_{X_i}^{-1}(v_i);\ \ i = 1, \ldots, n;\ \ \mathbf{v} \in [0,1]^n \ :\ C(\mathbf{v}) > \alpha\}.$$

Extremes through Copulas (Review)

Handle Copula Families & Marginal Distributions (G.E.V, etc)

↓

Sklar's Theorem

↓

Closed Multivariate Quantile Expressions

Drawbacks working with Copulas

This approach is hard to manipulate because:

- **The parametric nature generates complications and/or restrictions in high dimension, even using nested copula techniques.**
- **The models are quite rigid.**

DIRECTIONAL PROPOSAL OF EXTREMES IDENTIFICATION

Directional Multivariate Level-Sets

↓

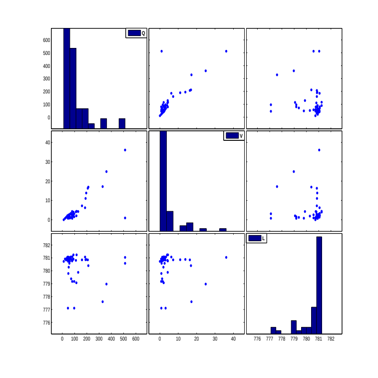
Directional Extremes

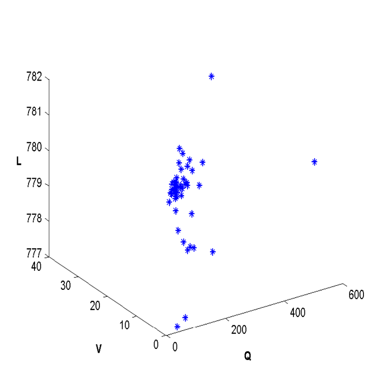
Looking the data in another direction

Why is useful a directional approach in environmental engineering?

To solve this question we simulate data from the model in Salvadori et al. (2011), which refers to data of maximum annual flood peaks Q, volumes V and water levels L in the Ceppo Morelli dam, Italy.

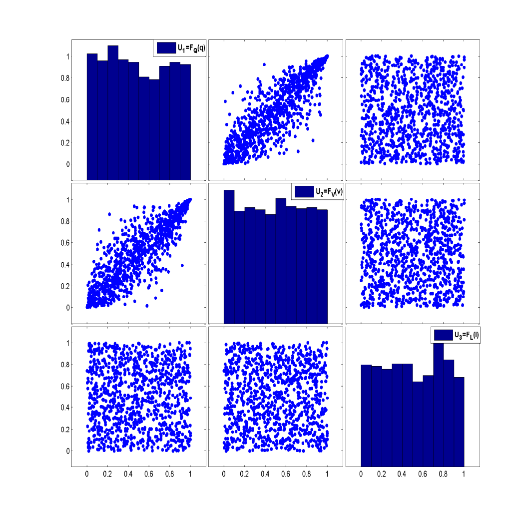
CEPPO MORELLI DAM EXAMPLE





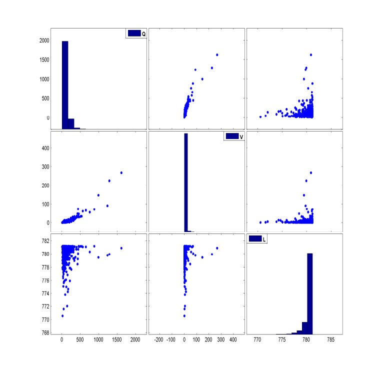
Original dataset of the Ceppo Morelli dam

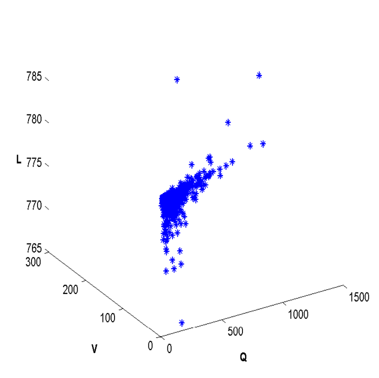
Ceppo Morelli dam example



Copula model for the Ceppo Morelli dam

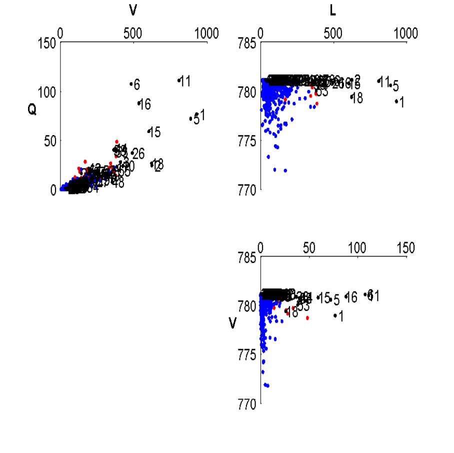
CEPPO MORELLI DAM EXAMPLE





Simulated Sample from the model in Salvadori et al.

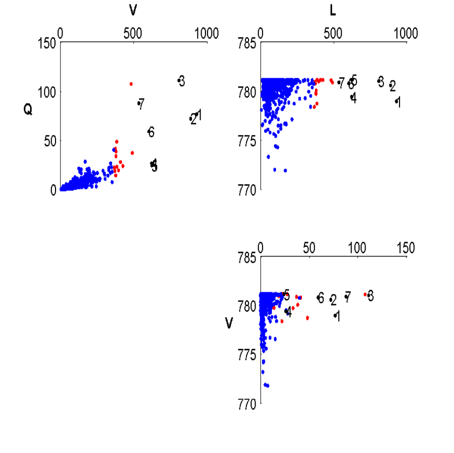
CEPPO MORELLI DAM EXAMPLE



Extremes with the non-parametric approach,

in the classic direction e for $\alpha = 1\%$

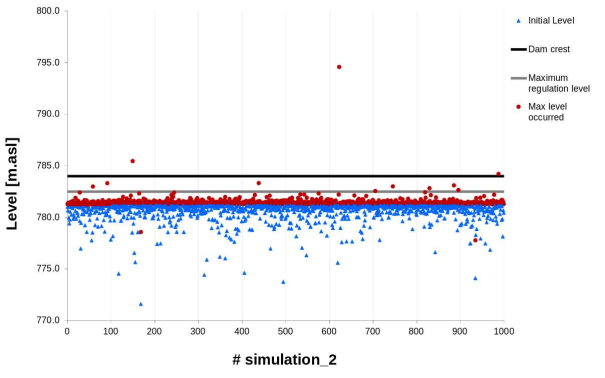
LOOKING THE DATA IN ANOTHER DIRECTION



Extremes with the non-parametric approach,

in the first *PCA* direction for $\alpha = 1\%$

DETERMINISTIC PHYSICAL DAM-LEVEL BEHAVIOR



Final level from the simulated occurrences of Q, V, L

Measures of false-positives and true-positives in classic and directional approaches

False-Positives

Classic Direction	PCA Direction
90%	35%

True-Positives

Classic Direction	PCA Direction
100%	100%

DIRECTIONAL MULTIVARIATE QUANTILES & COPULA APPROACH

THEOREM

Let $\mathbf{u}$ be fix, then the directional quantiles of a random vector $\mathbf{X}$ with *regularity conditions* are equivalent to those obtained by the copula procedure on the random vector $R_{\mathbf{u}}\mathbf{X}$, where $R_{\mathbf{u}}$ is the rotation matrix in the orthant definition.

Directional Multivariate Quantiles & Copula Approach

Sketch of the proof

Given α

$$\mathcal{Q}_{\mathbf{X}}(\alpha, -\mathbf{e}) \equiv F_{\mathbf{X}}^{-1}(\alpha) \& \mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{e}) \equiv \bar{F}_{\mathbf{X}}^{-1}(\alpha)$$

⇓

Under *regularity conditions*,

Orthogonal Quasi-Invariance & Sklar's Theorem over $R_{\pm\mathbf{u}}\mathbf{X}$

⇓

Directional approach $\equiv$ Copula approach.

Application in Financial Risk Measures

Let X be a random variable representing loss, F its distribution function and $0 \leq \alpha \leq 1$. Then,

$$VaR_\alpha(X) := \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\}.$$

Application in Financial Risk Measures

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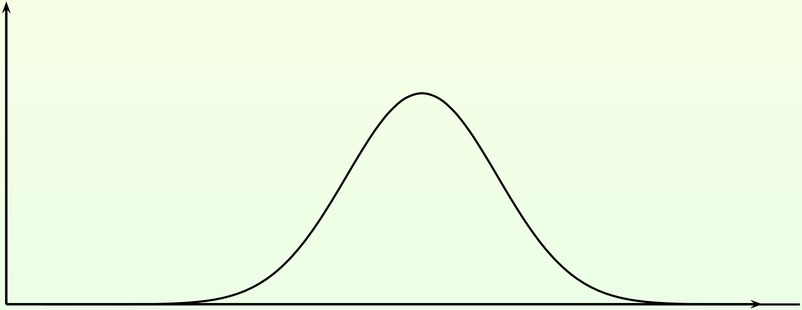

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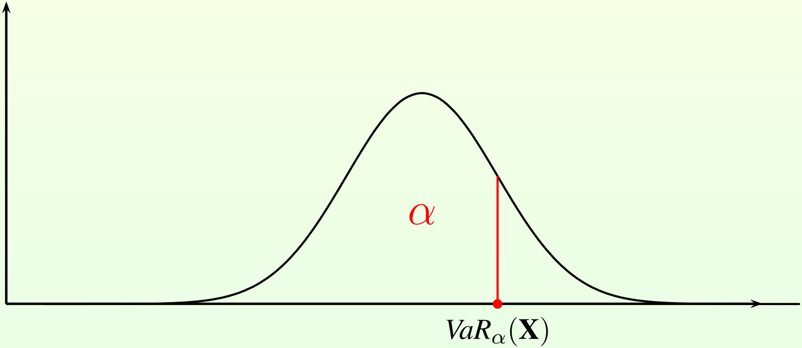

VALUE AT RISK (*VaR*)

- The *VaR* has became in a benchmark for risk management.
- The *VaR* has been criticized by Artzner et al. (1999) since it does not encourage diversification.
- But defended by Heyde et al. (2009) for its robustness and recently by Daníelsson et al. (2013) for its tail subadditivity.

MULTIVARIATE DRAWBACKS OF *VaR*

- **There is not a unique definition of a multivariate quantile.**
- **There are a lot of assets in a portfolio. (High Dimension)**
- **There is dependence among them.**

Review on Multivariate Value at Risk

An initial idea to study risk measures related to portfolios

$$\mathbf{X} = (X_1, \ldots, X_n),$$

is to consider a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ **and then:**

- **The *VaR* of the joint portfolio is the univariate-one associated to** $f(\mathbf{X})$**.**

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is to consider a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and then:

- **The *VaR* of the joint portfolio is the univariate-one associated to $f(\mathbf{X})$.**
- **In Burgert and Rüschendorf (2006),**

$$f(\mathbf{X}) = \sum_{i=1}^{n} X_i \text{ or } f(\mathbf{X}) = \max_{i \leq n} X_i.$$

Output: A NUMBER

Review on Multivariate Value at Risk

Embrechts and Puccetti (2006) introduced a multivariate approach of the Value at Risk,

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- **Multivariate lower-orthant Value at Risk**

$$\underline{VaR}_{\alpha}(\mathbf{X}) := \partial\{\mathbf{x} \in \mathbb{R}^n \mid F_{\mathbf{X}}(\mathbf{x}) \geq \alpha\}.$$

- **Multivariate upper-orthant Value at Risk**

$$\overline{VaR}_{\alpha}(\mathbf{X}) := \partial\{\mathbf{x} \in \mathbb{R}^n \mid \bar{F}_{\mathbf{X}}(\mathbf{x}) \leq 1 - \alpha\}.$$

Output: A SURFACE ON $\mathbb{R}^n$

Review on Multivariate Value at Risk

Cousin and Di Bernardino (2013) introduced a multivariate risk measure related to the measure introduced by Embrechts and Puccetti (2006).

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- **Multivariate upper-orthant Value at Risk**

$$\overline{VaR}_{\alpha}(\mathbf{X}) := \mathbb{E}\left[\mathbf{X} | \bar{F}_{\mathbf{X}}(\mathbf{x}) = 1 - \alpha\right].$$

Output: A point in $\mathbb{R}^n$

Directional Multivariate Value at Risk (*MVaR*)

Directional MVaR

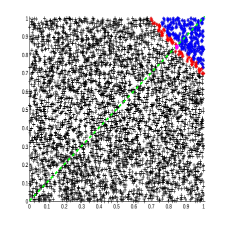
Let $\mathbf{X}$ be a random vector satisfying "*the regularity conditions*", then the Value at Risk of $\mathbf{X}$ in direction $\mathbf{u}$ and *confidence parameter* α is defined as

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) = \left(\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) \bigcap \{\lambda \mathbf{u} + \mathbb{E}[\mathbf{X}]\} \right),$$

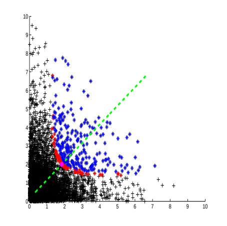
where $\lambda \in \mathbb{R}$ and $0 \leq \alpha \leq 1$.

Output: A POINT IN $\mathbb{R}^n$

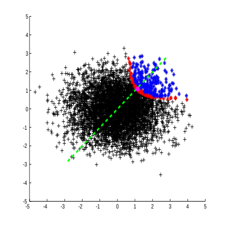
DIRECTIONAL MULTIVARIATE VALUE AT RISK (*MVaR*)



(A) Bivariate Uniform



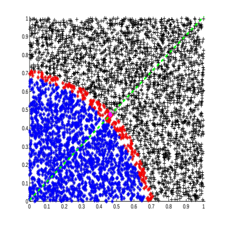
(B) Bivariate Exponential



(C) Bivariate Normal

$$VaR_{0.7}^{-\mathbf{e}}(\mathbf{X})$$

Directional Multivariate Value at Risk (*MVaR*)



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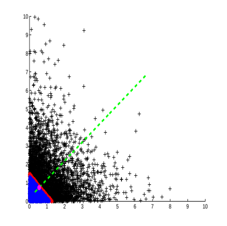

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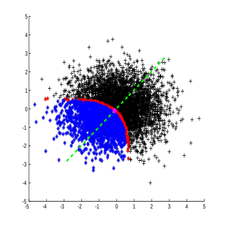


(C) Bivariate Normal

$$VaR^{\mathbf{e}}_{0.3}(\mathbf{X})$$

MVaR Properties

- **Non-Negative Loading**: If $\lambda > 0$,

$$\mathbb{E}[\mathbf{X}] \preceq_{\mathbf{u}} VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}),$$

where the order is given by

Preorder (Laniado et al. (2010))

$\mathbf{x}$ is said to be less than $\mathbf{y}$ if:

$$\mathbf{x} \preceq_{\mathbf{u}} \mathbf{y} \quad \equiv \quad \mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} \supseteq \mathfrak{C}_{\mathbf{y}}^{\mathbf{u}} \quad \equiv \quad R_{\mathbf{u}}\mathbf{x} \leq R_{\mathbf{u}}\mathbf{y}.$$

MVaR Properties

- **Quasi-Odd Measure**: $VaR_{\alpha}^{\mathbf{u}}(-\mathbf{X}) = -VaR_{\alpha}^{-\mathbf{u}}(\mathbf{X})$.

- **Positive Homogeneity and Translation Invariance**: Given $c \in \mathbb{R}^{+}$ and $\mathbf{b} \in \mathbb{R}^{n}$, then

$$VaR_{\alpha}^{\mathbf{u}}(c\mathbf{X} + \mathbf{b}) = cVaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) + \mathbf{b}.$$

MVaR Properties

- **Orthogonal Quasi-Invariance**: Let $\mathbf{w}$ and Q be an unit vector and a particular orthogonal matrix obtained by a QR decomposition such that $Q\mathbf{u} = \mathbf{w}$. Then,

$$VaR_{\alpha}^{\mathbf{w}}(Q\mathbf{X}) = QVaR_{\alpha}^{\mathbf{u}}(\mathbf{X}).$$

MVaR Properties

- **Consistency**: Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors such that $\mathbb{E}[\mathbf{Y}] = c\mathbf{u} + \mathbb{E}[\mathbf{X}]$, for $c > 0$ and $\mathbf{X} \leq_{\mathcal{E}_\mathbf{u}} \mathbf{Y}$. Then:

$$VaR_\alpha^\mathbf{u}(\mathbf{X}) \preceq_\mathbf{u} VaR_\alpha^\mathbf{u}(\mathbf{Y}),$$

where the stochastic order is defined by

Stochastic Extremality Order (Laniado et al. (2012))

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors in $\mathbb{R}^n$,

$$\mathbf{X} \leq_{\mathcal{E}_\mathbf{u}} \mathbf{Y} \quad \equiv \quad \mathbb{P}\left[R_\mathbf{u}(\mathbf{X}-\mathbf{z}) \geq 0\right] \leq \mathbb{P}\left[R_\mathbf{u}(\mathbf{Y}-\mathbf{z}) \geq 0\right] \quad \equiv \quad \mathbb{P}_\mathbf{X}\left[\mathfrak{C}_\mathbf{z}^\mathbf{u}\right] \leq \mathbb{P}_\mathbf{Y}\left[\mathfrak{C}_\mathbf{z}^\mathbf{u}\right],$$

for all $\mathbf{z}$ in $\mathbb{R}^n$.

MVaR Properties

- **Non-Excessive Loading**: For all $\alpha \in (0,1)$ and $\mathbf{u} \in \mathbb{B}(0)$,

$$VaR_\alpha^{\mathbf{u}}(\mathbf{X}) \preceq_{\mathbf{u}} R'_{\mathbf{u}} \sup_{\omega \in \Omega} \{R_{\mathbf{u}}\mathbf{X}(\omega)\}.$$

- **Subadditivity in the Tail Region**: Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors, with the same mean $\boldsymbol{\mu}$ and let $(R_{\mathbf{u}}\mathbf{X}, R_{\mathbf{u}}\mathbf{Y})$ be a regularly varying random vector. Then,

$$VaR_\alpha^{\mathbf{u}}(\mathbf{X}+\mathbf{Y}) \preceq_{\mathbf{u}} VaR_\alpha^{\mathbf{u}}(\mathbf{X}) + VaR_\alpha^{\mathbf{u}}(\mathbf{Y}).$$

Lower and Upper Versions of Directional *MVaR*

Result

Let $\mathbf{X}$ be a random vector and $\mathbf{u}$ a direction. Then for all $0 \leq \alpha \leq 1$,

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}) \preceq_{\mathbf{u}} VaR_{1-\alpha}^{-\mathbf{u}}(\mathbf{X}).$$

Lower and Upper Versions of Directional *MVaR*

Then, analogously as Embrechts and Puccetti (2006) and Cousin and Di Bernardino (2013), we can define:

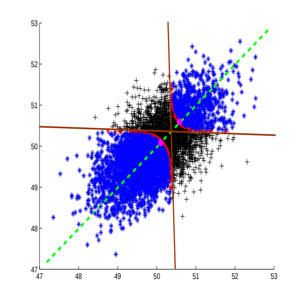
***Lower Multivariate VaR* in the direction $\mathbf{u}$ as**

$$VaR_{\alpha}^{\mathbf{u}}(\mathbf{X}),$$

***Upper Multivariate VaR* in the direction $\mathbf{u}$ as**

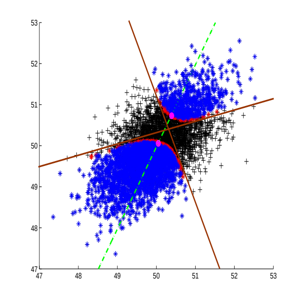
$$VaR_{1-\alpha}^{-\mathbf{u}}(\mathbf{X}).$$

Lower and Upper Versions of Directional *MVaR*



Lower Multivariate VaR $= VaR^{\mathbf{e}}_{0.3}(\mathbf{X})$ and
Upper Multivariate VaR $= VaR^{-\mathbf{e}}_{0.7}(\mathbf{X})$

Lower and Upper Versions of Directional *MVaR*



Lower Multivariate VaR $= VaR_{0.3}^{(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})}(\mathbf{X})$ and

Upper Multivariate VaR $= VaR_{0.7}^{-(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})}(\mathbf{X})$

Relation between the marginal *VaR* and the *MVaR*

Result

Let $\mathbf{X}$ be a random vector with survival function $\bar{F}$ quasi-concave. Then, for all $\alpha \in (0,1)$:

$$VaR_{1-\alpha}(X_i) \geq \left[VaR_{\alpha}^{\mathbf{e}}(\mathbf{X})\right]_i, \qquad \text{for all} \quad i = 1, ..., n.$$

Moreover, if its distribution function F is quasi-concave, then, for all $\alpha \in (0,1)$,

$$\left[VaR_{1-\alpha}^{-\mathbf{e}}(\mathbf{X})\right]_i \geq VaR_{1-\alpha}(X_i), \qquad \text{for all} \quad i = 1, ..., n.$$

RELATION BETWEEN THE MARGINAL *VaR* AND THE *MVaR*

RESULT

Let $\mathbf{X}$ be a random vector and $\mathbf{u}$ a direction. If the survival function of $R_{\mathbf{u}}\mathbf{X}$ is quasi-concave. Then, for all $0 \le \alpha \le 1$,

$$VaR_{1-\alpha}([R_{\mathbf{u}}\mathbf{X}]_i) \ge [R_{\mathbf{u}} VaR^{\mathbf{u}}_{\alpha}(\mathbf{X})]_i \,, \qquad \text{for all} \quad i = 1, ..., n.$$

And if $R_{\mathbf{u}}X$ has a quasi-concavity cumulative distribution, we have that

$$\left[R_{\mathbf{u}} VaR^{-\mathbf{u}}_{1-\alpha}(\mathbf{X})\right]_i \ge VaR_{1-\alpha}([R_{\mathbf{u}}X]_i), \qquad \text{for all} \quad i = 1, ..., n.$$

ROBUSTNESS

We analyze the behavior of the *MVaR* when a sample is contaminated with different types of outliers.

We use as a benchmark the measurement given by the multivariate *VaR* in Cousin and Di Bernardino (2013).

Robustness

We simulate 5000 observations of the following random vector:

$$\mathbf{X}^{\omega} \overset{\mathbf{d}}{=} \begin{cases} \mathbf{X}_1 & \text{with probability } p = 1-\omega, \\ \mathbf{X}_2 & \text{with probability } p = \omega, \end{cases}$$

where $\mathbf{X}_1 \overset{\mathbf{d}}{=} N_1(\boldsymbol{\mu}_1, \Sigma_1)$**,** $\mathbf{X}_2 \overset{\mathbf{d}}{=} N_2(\boldsymbol{\mu}_1 + \Delta_{\boldsymbol{\mu}}, \Sigma_1 + \Delta_{\boldsymbol{\Sigma}})$ **and** $0 \leq \omega \leq 1$**. Specifically:**

$$\boldsymbol{\mu}_1 = [50, 50]', \qquad \Sigma_1 = \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{pmatrix}.$$

Contaminating
1. **Varying only the mean.**
2. **Varying only the variances.**
3. **Varying all the parameters.**

Robustness

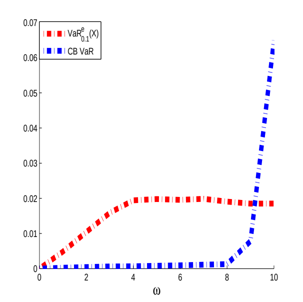
To evaluate the impact of the contamination, we use:

$$PV^{\omega} = \frac{||Measure(\mathbf{X}^{\omega}) - Measure(\mathbf{X}^{0})||_2}{||Measure(\mathbf{X}^{0})||_2},$$

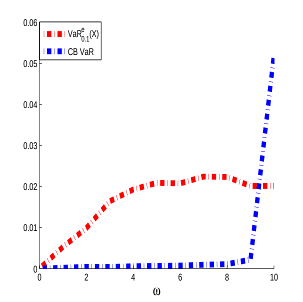
where $Measure(\mathbf{X}^0)$ **is the sample with** $\omega = 0\%$ **and** $Measure(\mathbf{X}^{\omega})$ **is the sample with level of contamination** $\omega\%$**, (**$\omega = 1\% \to 10\%$**).**

ROBUSTNESS

1. Varying only the mean, $\Delta_{\boldsymbol{\mu}} \neq 0, \quad \Delta_{\boldsymbol{\Sigma}} = 0$.



(A) $\Delta_{\boldsymbol{\mu}} = (20, 20)'$

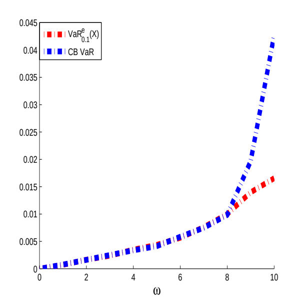


(B) $\Delta_{\boldsymbol{\mu}} = (0, 50)'$

Mean of PV^{ω}

ROBUSTNESS

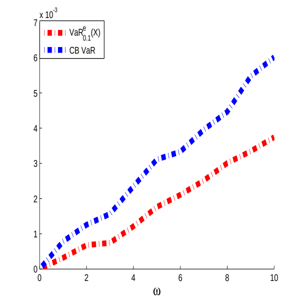
2. Varying only the variances, $\Delta_{\boldsymbol{\mu}} = 0, \quad \Delta_{\boldsymbol{\Sigma}} = \begin{bmatrix} 4.5 & 0 \\ 0 & 6.5 \end{bmatrix}$,



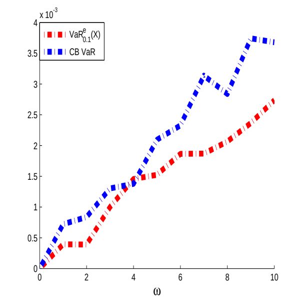
Mean of PV^{ω}

ROBUSTNESS

3. Varying all the parameters, $\Delta_{\boldsymbol{\mu}} \neq 0, \quad \Delta_{\boldsymbol{\Sigma}} = \begin{bmatrix} 4.5 & 0.2 \\ 0.3 & 6.5 \end{bmatrix},$



(A) $\Delta_{\boldsymbol{\mu}} = (20, 20)'$



(B) $\Delta_{\boldsymbol{\mu}} = (0, 50)'$

Mean of PV^{ω}

Current Research about Estimation

Result

Let $\mathbf{X}$ be a multivariate regularly varying random vector with tail index β. Then for all orthogonal transformation Q the random vector $Q\mathbf{X}$ is regularly varying with tail index β.

Current Research about Estimation

Using as base the theory in De Haan and Huang (1995) for a bivariate estimation of quantile curves, we want to:

- Study the extension to higher dimensions.

CURRENT RESEARCH ABOUT ESTIMATION

Using as base the theory in De Haan and Huang (1995) for a bivariate estimation of quantile curves, we want to:

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- **Link the directional notion into the theory by using previous result.**

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- Study convergence and consistency, theoretically and practically.

CURRENT RESEARCH ABOUT ESTIMATION

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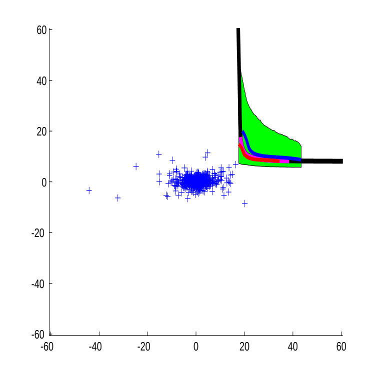
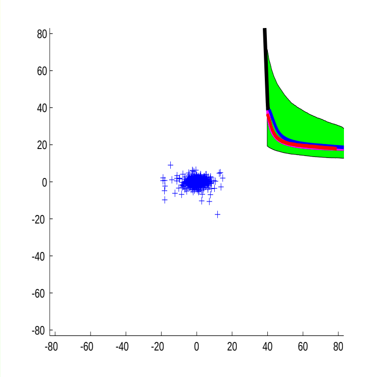
- **Study the extension to higher dimensions.**
- **Link the directional notion into the theory by using previous result.**
- **Study convergence and consistency, theoretically and practically.**
- **Make comparisons between the previous non-parametric approach and the resultant by this research.**

MULTIVARIATE HEAVY TAILED EXAMPLE



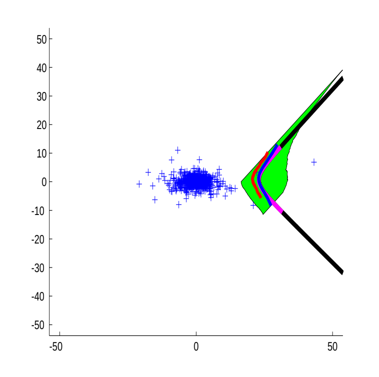
(A) $n = 500, \quad \alpha = \frac{1}{n}$



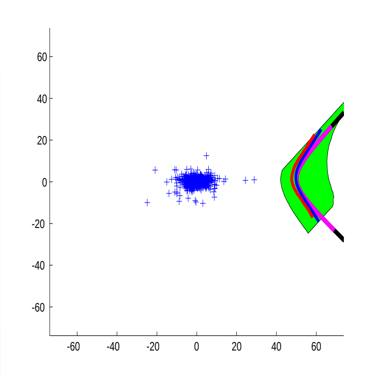
(B) $n = 5000, \quad \alpha = \frac{1}{n}$

Bivariate $t-$distribution with $\nu = 3$

Estimation in the first PCA direction



(A) $n = 500, \quad \alpha = \frac{1}{n}$



(B) $n = 5000, \quad \alpha = \frac{1}{n}$

Bivariate $t-$distribution with $\nu = 3$

Conclusions and Future Research

- We have introduced an extension of the multivariate extreme value analysis by introducing a directional notion.

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- We have introduced an extension of the multivariate extreme value analysis by introducing a directional notion.
- The directional approach allows to consider external intrinsic information of a system or management preferences in the analysis.
- We provide arguments of the needing of these directional approach in practice. As well as, we present theoretical properties and results of this directional extension in the developed applications.
- Two important aspects have been studied. Asymptotic convergence and robustness in more general cases.

Thanks

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