Ensuring robustness of domain decomposition methods by block strategies

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Motivations

FETI, BDD and similar non-overlapping DD's

- Mature solution techniques to enable computing the solution of large systems on parallel computers (Charbel Farhat et al., 1991; Jan Mandel, 1993).
- The fundamental idea: solving local problems related to each domain with techniques that perform well sequentially on one processor and applying iterative techniques to find the interface unknowns connecting domains together.

Practical difficulties

Some situations lead to less efficiency

- High heterogeneity (C. Farhat et al., 1995; Klawonn et al., 2007; D. Rixen et al., 1999).
- Bad aspect ratio of subdomains
- Jagged interfaces (Klawonn et al., 2008).
- Incompressibility (Vereecke et al., 2003).

these difficulties prevent a massive adoption of the method by industrialists

Motivations

Geneo: Generalized Eigenvalues in the Overlaps

- In (Nicole Spillane et al., 2013; N. Spillane et al., 2014) the bad convergence of domain decomposition strategies is traced back to the fact that important global characteristics of the global problem can not be approximated by the local information typically used to precondition the iterations on the interface problem.
- Generalized eigenvalue problems between subdomains can reveal the problematic modes (prior analysis).
- In (Nicole Spillane et al., 2013; N. Spillane et al., 2014) deflation strategies (or coarse grid approaches) are applied to guarantee that the iterations are performed only on the part of the space that can be properly preconditioned.

This method cures almost all problems but with a significant computational overhead related to finding the coarse space of "bad" modes through eigenvalue problems.

Illustration

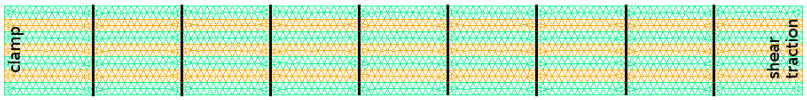


Figure: Beam with straight decomposition

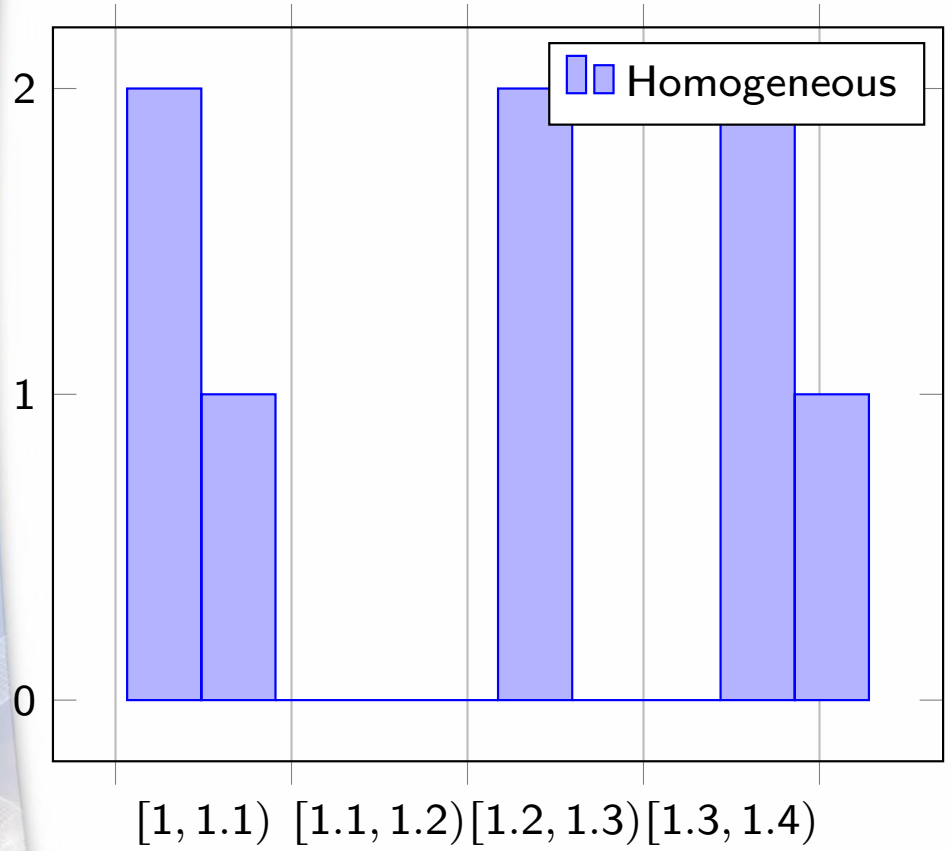


Figure: Spectrum

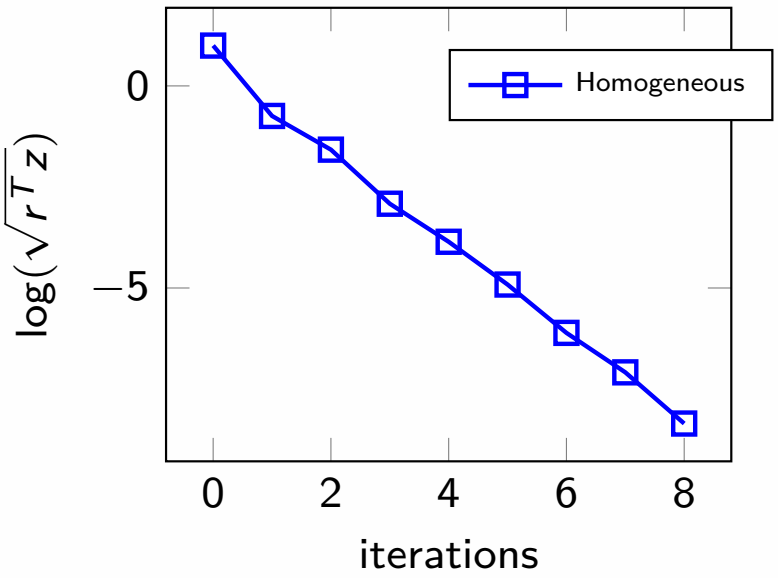


Figure: Convergence

Illustration

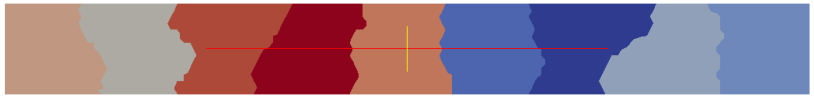


Figure: Beam with irregular decomposition

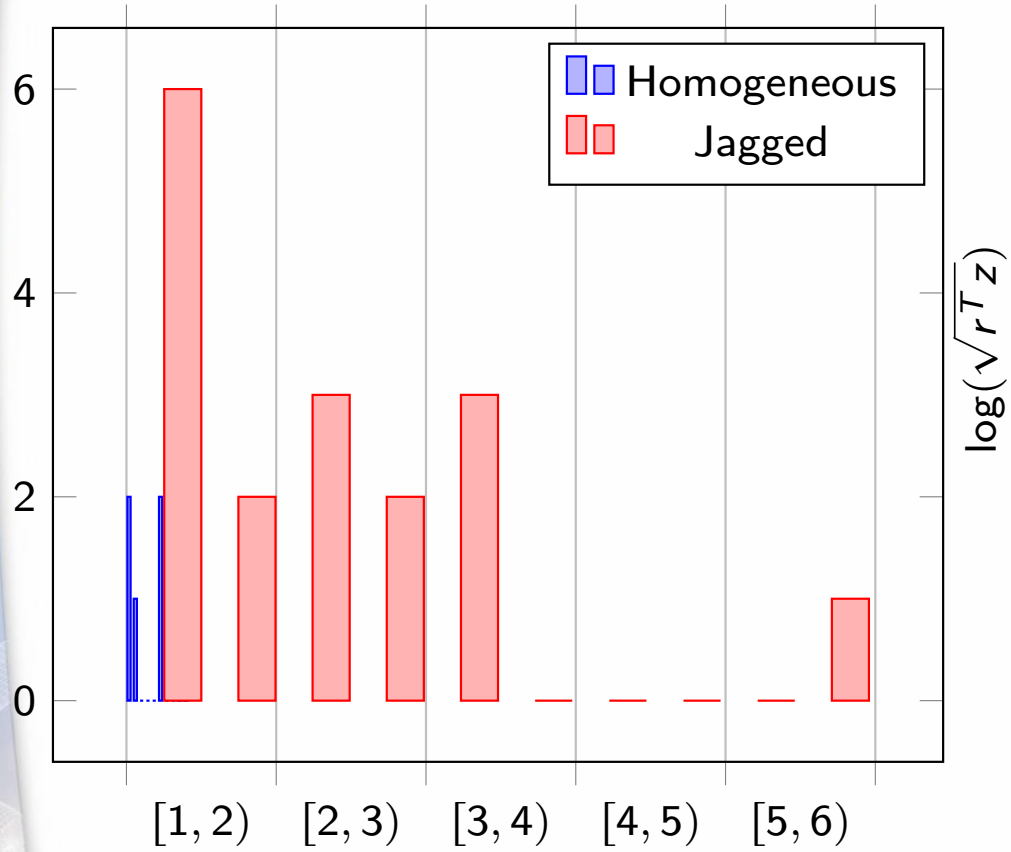


Figure: Spectrum

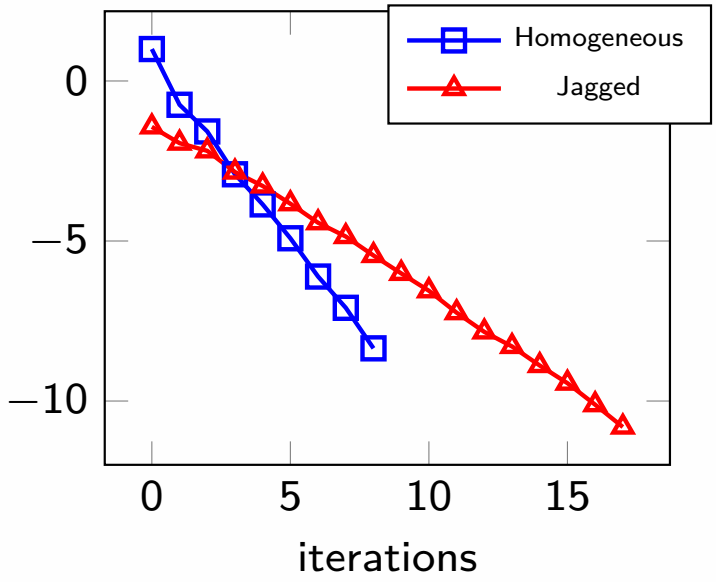


Figure: Convergence

Illustration

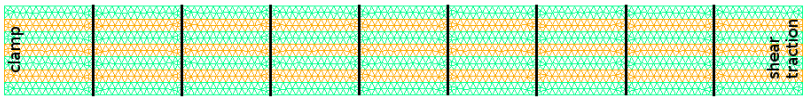
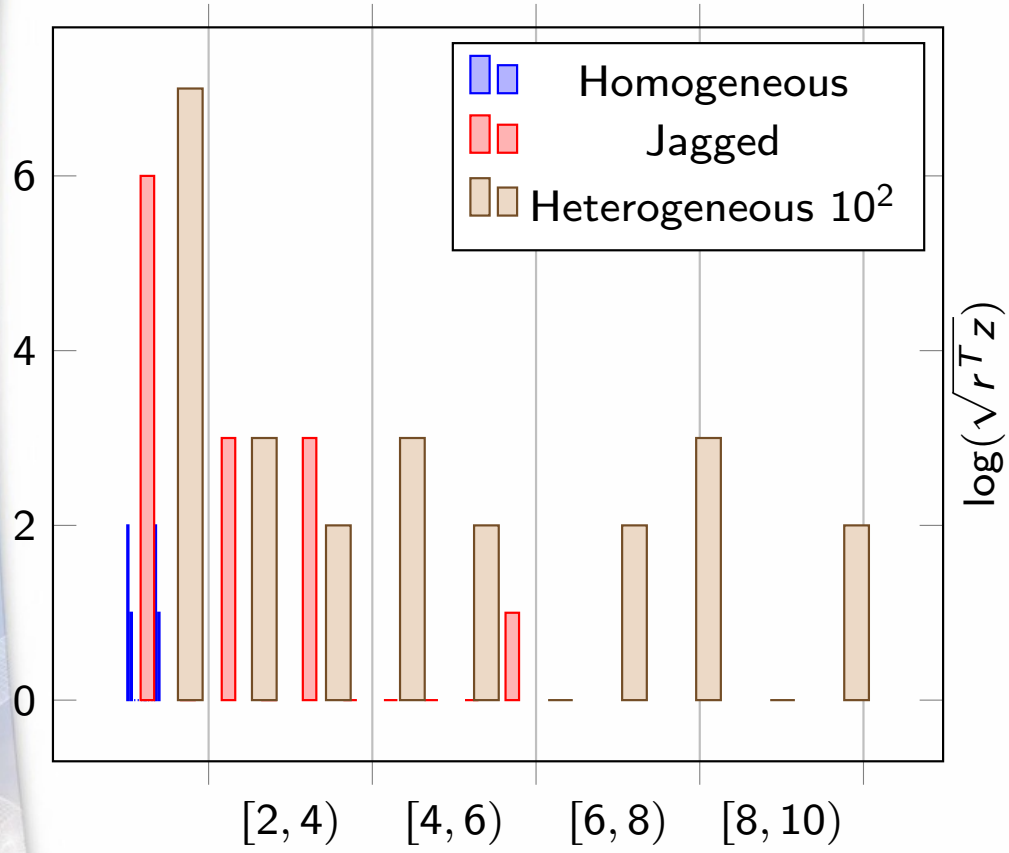


Figure: Beam with straight decomposition



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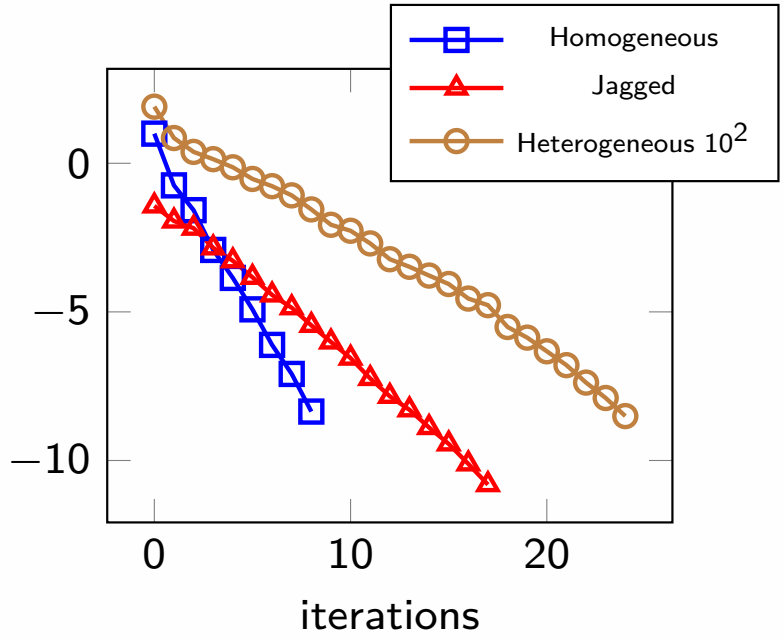

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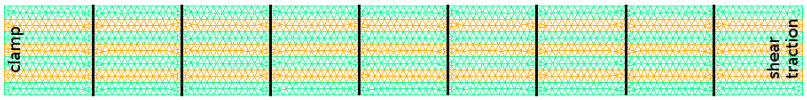

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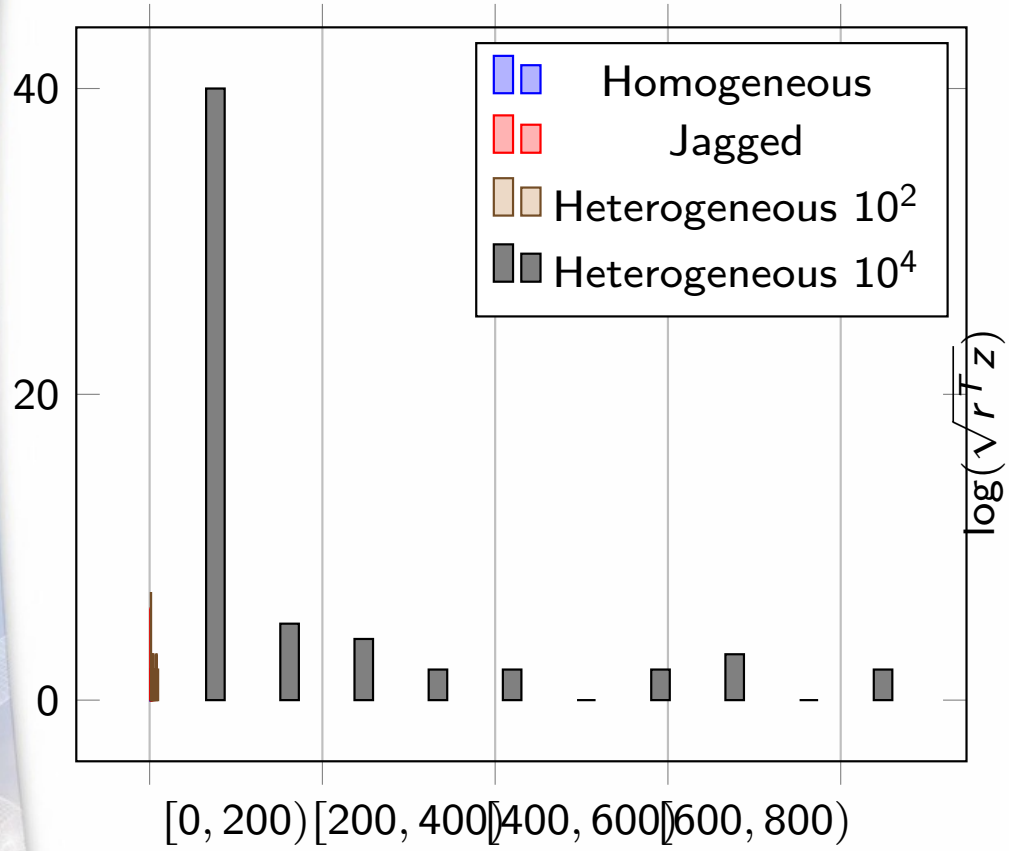

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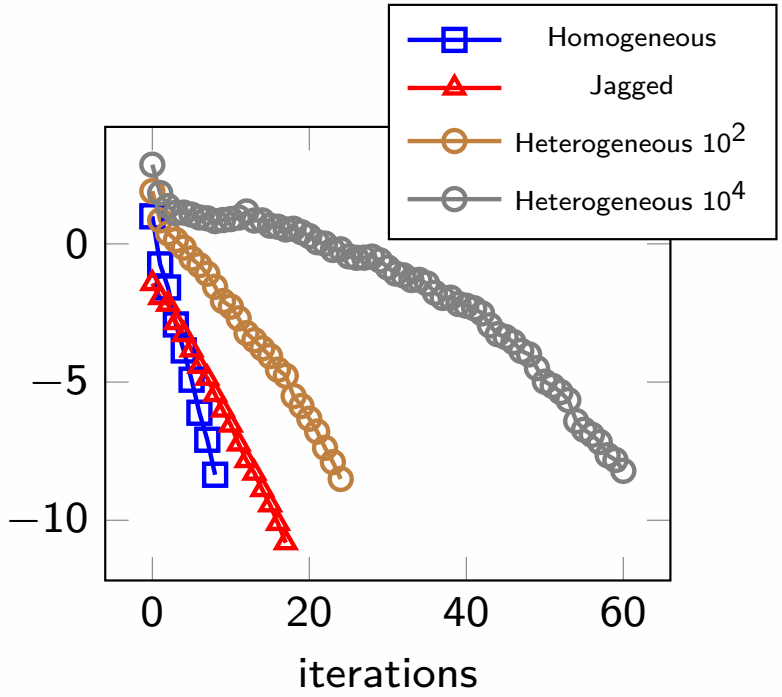

Figure: Convergence

Objectives

We present two strategies to achieve performance similar to GENEO without the a priori computation: we want to detect and deal with critical local contributions on the fly.

Note that although only the FETI approaches are discussed here, the ideas presented in this contribution can be extended in a straightforward manner to the other variants of FETI (such as the FETI-DP method (C. Farhat, Lesoinne, et al., 2001; C. Farhat et al., 2000)) or to primal Schur complement methods such as BDD.

Our conjecture is:

$$n_{block} \leqslant n_{geneo} + \max(n_{lanczos})$$

n_{block} number of iterations for bloc solvers
n_{geneo} number of iterations for Geneo solver
$n_{lanczos}$ number of iterations to compute "bad" eigenvectors

Outlook

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6. Conclusion

Global system on domain Ω

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

$\mathbf{K}$ is SPD

Substructured formulation

N non-overlapping subdomains, conforming mes)

$$\mathbf{K}^{(s)}\mathbf{u}^{(s)} = \mathbf{f}^{(s)} + \mathbf{B}^{(s)^T}\mathbf{t}^{(s)^T}\boldsymbol{\lambda}$$

$$\sum_s \mathbf{B}^{(s)}\mathbf{t}^{(s)}\mathbf{u}^{(s)} = 0$$

$\boldsymbol{\lambda}$ = Lagrange multipliers that connect subdomains

Classical FETI system

$$\begin{pmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{d} \\ \mathbf{e} \end{pmatrix}$$

Topology

$\mathbf{t}^{(s)}$ = trace operators
$\mathbf{B}^{(s)}$ = signed Boolean assembly operators

Local operators

$\mathbf{S}^{(s)} = \mathbf{K}_{bb}^{(s)} - \mathbf{K}_{bi}^{(s)}\mathbf{K}_{ii}^{(s)^{-1}}\mathbf{K}_{ib}^{(s)}$ Schur complement
$\mathbf{F}^{(s)} = (\mathbf{S}^{(s)})^+ = \mathbf{t}^{(s)}\mathbf{K}^{(s)^+}\mathbf{t}^{(s)^T}$ Dual Schur
$\mathbf{R}^{(s)} = \ker(\mathbf{K}^{(s)})$ basis of rigid body modes

Global operators

$$\mathbf{e} = -\left(\ldots, \mathbf{f}^{(s)^T}\mathbf{R}^{(s)}, \ldots\right)^T$$

$$\mathbf{G} = \left(\ldots, \mathbf{B}^{(s)}\mathbf{t}^{(s)}\mathbf{R}^{(s)}, \ldots\right)$$

$$\mathbf{F} = \sum_s \mathbf{B}^{(s)}\mathbf{F}^{(s)}\mathbf{B}^{(s)^T}$$

$$\mathbf{d} = -\sum_s \mathbf{B}^{(s)}\mathbf{t}^{(s)}\mathbf{K}^{(s)^+}\mathbf{f}^{(s)}$$

Rigid body constraint

$$\boldsymbol{\lambda}_0 = \mathbf{AG}(\mathbf{G}^T\mathbf{AG})^{-1}\mathbf{e}$$
$$\mathbf{P} = \mathbf{I} - \mathbf{AG}(\mathbf{G}^T\mathbf{AG})^{-1}\mathbf{G}^T$$

Matrix $\mathbf{A}$ is a SPD matrix, $\mathbf{A} \simeq \tilde{\mathbf{S}}$ approximates the preconditioner (D. J. Rixen et al., 1999).

$\boldsymbol{\lambda}$ is sought as $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \mathbf{P}\tilde{\boldsymbol{\lambda}}$ where $\tilde{\boldsymbol{\lambda}}$ is a solution of:

$$\mathbf{P}^T\mathbf{FP}\tilde{\boldsymbol{\lambda}} = \mathbf{P}^T(\mathbf{d} - \mathbf{F}\boldsymbol{\lambda}_0) = \mathbf{P}^T\left(\sum_s \mathbf{B}^{(s)}\mathbf{K}^{(s)^+}(\mathbf{f}^{(s)} - \mathbf{B}^{(s)^T}\boldsymbol{\lambda}_0)\right)$$

Dirichlet preconditioner

This system is solved by an iterative solver, the preconditioner $\tilde{\mathbf{S}}$ being

$$\tilde{\mathbf{S}} = \sum_s \tilde{\mathbf{B}}^{(s)}\tilde{\mathbf{S}}^{(s)}\tilde{\mathbf{B}}^{(s)^T}$$

$\tilde{\mathbf{B}}^{(s)}$ are scaled assembling operators / $\sum_s \mathbf{B}^s\tilde{\mathbf{B}}^{(s)^T} = \mathbf{I}$ (D. Rixen et al., 1999)
$\tilde{\mathbf{S}}^{(s)}$ are the Schur complements $\mathbf{S}^{(s)}$ or an approximation

Extra constraint (FETI2) (C. Farhat et al., 1998; C. Farhat et al., 1998)

$\mathbf{C}^T\mathbf{r} = 0$ where matrix $\mathbf{C}$ is a basis of a well-chosen subspace and $\mathbf{r}$ is the residual. Initialization/projection:

$$\tilde{\boldsymbol{\lambda}}_0 = \mathbf{C}(\mathbf{C}^T\mathbf{F}\mathbf{C})^{-1}\mathbf{C}^T(\mathbf{d} - \mathbf{F}\boldsymbol{\lambda}_0)$$
$$\mathbf{P}_\mathbf{C} = \mathbf{I} - \mathbf{C}(\mathbf{C}^T\mathbf{F}\mathbf{C})^{-1}\mathbf{C}^T\mathbf{F}$$

Ideal $\mathbf{C}$

Eigenvectors associated to the largest eigenvalues of the following generalized system:

$$\mathbf{P}^T\left(\mathbf{F}\mathbf{v} - \mu\tilde{\mathbf{S}}^{-1}\mathbf{v}\right) = 0, \quad \mathbf{v} \in \text{range}(\mathbf{P})$$

Approximations by recycling of nearby Krylov subspaces (Gosselet et al., 2013).

Geneo result

(Nicole Spillane et al., 2013) these eigenvectors always originate from local effects so that they can be generated by a family of local eigenvalue problems:

$$\mathbf{S}^{(s)}\mathbf{v}^{(s)} - \mu^{(s)}{\mathbf{B}^{(s)}}^T\tilde{\mathbf{S}}\mathbf{B}^{(s)}\mathbf{v}^{(s)} = 0$$
$$\mathbf{C} = \{\mathbf{P}\tilde{\mathbf{S}}\mathbf{B}^{(s)}\mathbf{v}^{(s)};\ 0 < \mu^{(s)} < \tau\}$$

FETI Algorithm

Algorithm 1: FETI2 with full orthogonalization

$\mathbf{r}_0 = \mathbf{P}^T \mathbf{P_C}^T (\mathbf{d} - \mathbf{F}\boldsymbol{\lambda}_0)$

$\mathbf{z}_0 = \tilde{\mathbf{S}}\mathbf{r}$, $\mathbf{w}_0 = \mathbf{P}\mathbf{z}_0$, $\hat{\boldsymbol{\lambda}}_0 = 0$, $i = 0$

while $\sqrt{\mathbf{r}_i^T \mathbf{z}_i} > \epsilon$ **do**

$\mathbf{q}_i = \mathbf{P_C}^T \mathbf{F}\mathbf{w}_i$

$\delta_i = \mathbf{q}_i^T \mathbf{w}_i$

$\gamma_i = \mathbf{r}_i^T \mathbf{z}_i$

$\hat{\boldsymbol{\lambda}}_{i+1} = \hat{\boldsymbol{\lambda}}_i + (\gamma_i/\delta_i)\mathbf{w}_i$

$\mathbf{r}_{i+1} = \mathbf{r}_i - (\gamma_i/\delta_i)\mathbf{P}^T \mathbf{q}_i$

$\mathbf{z}_{i+1} = \tilde{\mathbf{S}}\mathbf{r}_{i+1}$

$\mathbf{w}_{i+1} = \mathbf{P}\mathbf{z}_{i+1}$ then for $0 \leqslant j \leqslant i$ $\begin{cases} \phi_{i,j} = \mathbf{q}_j^T \mathbf{w}_{i+1} \\ \mathbf{w}_{i+1} \leftarrow \mathbf{w}_{i+1} - (\phi_{i,j}/\delta_j)\mathbf{w}_j \end{cases}$

$i \leftarrow i + 1$

end

$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \tilde{\boldsymbol{\lambda}}_0 + \mathbf{P_C}\hat{\boldsymbol{\lambda}}_i$

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Simultaneous FETI

S-FETI was introduced in (D. Rixen, 1997) on an example with two subdomains. It exploits the additive structure of the preconditioner in order to generate as many search directions as there are subdomains at each step of CG.

Classical FETI preconditioned residual

$$\mathbf{z} = \tilde{\mathbf{S}}\mathbf{r} = \sum_s \widetilde{\mathbf{B}}^{(s)} \tilde{\mathbf{S}}^{(s)} \widetilde{\mathbf{B}}^{(s)^T} \mathbf{r}$$

S- FETI preconditioned residual

The idea underlying the S-FETI approach consists in letting the minimization process of the Conjugate Gradient choose what the best combination should be. To obtain $\mathbf{z}$ each local term $\widetilde{\mathbf{B}}^{(s)} \tilde{\mathbf{S}}^{(s)} \widetilde{\mathbf{B}}^{(s)^T}$ is used as a search direction, namely the residual is minimized with respect to the subspace spanned by

$$\mathbf{Z} = \left(\ldots, \widetilde{\mathbf{B}}^{(s)} \tilde{\mathbf{S}}^{(s)} \widetilde{\mathbf{B}}^{(s)^T} \mathbf{r}, \ldots \right)$$

Connection

$\mathbf{z} = \mathbf{Z}\mathbf{1}$ where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^N$ which enables us to monitor convergence in a classical way.

Algorithm 2: Simultaneous FETI

$\mathbf{r}_0 = \mathbf{P}^T(\mathbf{d} - \mathbf{F}\boldsymbol{\lambda}_0)$

$\mathbf{Z}_0 = \left(\ldots, \widetilde{\mathbf{B}}^{(s)}\mathbf{S}^{(s)}\mathbf{r}_0, \ldots\right)$, $\mathbf{W}_0 = \mathbf{P}\mathbf{Z}_0$

$\tilde{\boldsymbol{\lambda}}_0 = 0$, $i = 0$

while $\sqrt{\mathbf{r}^T\mathbf{Z}\mathbf{1}} > \epsilon$ **do**

$\mathbf{Q}_i = \mathbf{F}\mathbf{W}_i$

$\boldsymbol{\Delta}_i = \mathbf{Q}_i^T\mathbf{W}_i$

$\gamma_i = \mathbf{Z}_i^T\mathbf{r}_i$

$\tilde{\boldsymbol{\lambda}}_{i+1} = \tilde{\boldsymbol{\lambda}}_i + \mathbf{W}_i\boldsymbol{\Delta}_i^+\gamma_i$

$\mathbf{r}_{i+1} = \mathbf{r}_i - \mathbf{P}^T\mathbf{Q}_i\boldsymbol{\Delta}_i^+\gamma_i$

$\mathbf{Z}_{i+1} = \left(\ldots, \widetilde{\mathbf{B}}^{(s)}\mathbf{S}^{(s)}\widetilde{\mathbf{B}}^{(s)^T}\mathbf{r}_{i+1}, \ldots\right)$

$\mathbf{W}_{i+1} = \mathbf{P}\mathbf{Z}_{i+1}$

for $0 \leqslant j \leqslant i$

$\boldsymbol{\Phi}_{i,j} = \mathbf{Q}_j^T\mathbf{W}_{i+1}$

$\mathbf{W}_{i+1} \leftarrow \mathbf{W}_{i+1} - \mathbf{W}_j\boldsymbol{\Delta}_j^+\boldsymbol{\Phi}_{i,j}$

$i \leftarrow i + 1$

end

$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \tilde{\boldsymbol{\lambda}}_i$

Algorithm 3: FETI2

$\mathbf{r}_0 = \mathbf{P}^T{\mathbf{P}_\mathbf{C}}^T(\mathbf{d} - \mathbf{F}\boldsymbol{\lambda}_0)$

$\mathbf{z}_0 = \tilde{\mathbf{S}}\mathbf{r}$, $\mathbf{w}_0 = \mathbf{P}\mathbf{z}_0$

$\hat{\boldsymbol{\lambda}}_0 = 0$, $i = 0$

while $\sqrt{\mathbf{r}_i^T\mathbf{z}_i} > \epsilon$ **do**

$\mathbf{q}_i = {\mathbf{P}_\mathbf{C}}^T\mathbf{F}\mathbf{w}_i$

$\delta_i = \mathbf{q}_i^T\mathbf{w}_i$

$\gamma_i = \mathbf{r}_i^T\mathbf{z}_i$

$\hat{\boldsymbol{\lambda}}_{i+1} = \hat{\boldsymbol{\lambda}}_i + (\gamma_i/\delta_i)\mathbf{w}_i$

$\mathbf{r}_{i+1} = \mathbf{r}_i - (\gamma_i/\delta_i)\mathbf{P}^T\mathbf{q}_i$

$\mathbf{z}_{i+1} = \tilde{\mathbf{S}}\mathbf{r}_{i+1}$

$\mathbf{w}_{i+1} = \mathbf{P}\mathbf{z}_{i+1}$

for $0 \leqslant j \leqslant i$

$\phi_{i,j} = \mathbf{q}_j^T\mathbf{w}_{i+1}$

$\mathbf{w}_{i+1} \leftarrow \mathbf{w}_{i+1} - (\phi_{i,j}/\delta_j)\mathbf{w}_j$

$i \leftarrow i + 1$

end

$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \tilde{\boldsymbol{\lambda}}_0 + \mathbf{P}_\mathbf{C}\hat{\boldsymbol{\lambda}}_i$

Cost of S-FETI

Sizes of blocks (n number of unknowns, N number of subdomains)

$$\mathbf{r}_i,\ \tilde{\boldsymbol{\lambda}}_i,\ \boldsymbol{\lambda},\ \boldsymbol{\lambda}_0,\ \in \mathbb{R}^n; \quad \mathbf{Z}_i,\ \mathbf{W}_i,\ \mathbf{Q}_i\ \in \mathbb{R}^{n\times N}; \quad \boldsymbol{\Delta}_i,\ \boldsymbol{\Phi}_{i,j}\ \in \mathbb{R}^{N\times N}; \quad \boldsymbol{\gamma}_i \in \mathbb{R}^N \quad ;$$

Costs

- Exchanges are as frequent. Global reduction operations (due to scalar products) involve more data.
- Dense, but usually small $N \times N$ symmetric positive matrices $\boldsymbol{\Delta}$ need to be (pseudo)-inversed. In any case the right-hand-sides $\boldsymbol{\gamma}_i$, $\boldsymbol{\Phi}_{i,j}$ are both in $\text{range}(\mathbf{W}_i^T) = \text{range}(\boldsymbol{\Delta}_i)$ so that the iteration is always well defined.
- Sequences of N-blocks of vectors $(\mathbf{W}_i)$, $(\mathbf{Q}_i)$ need to be stored instead of sequences of vectors.
- The computation of $\mathbf{FW}_i$ is a block operation moreover since

$$\mathbf{Q}_{i+1} = \mathbf{FW}_{i+1} = \left(\mathbf{FZ}_{i+1} - \mathbf{FAG}(\mathbf{G}^T\mathbf{AG})^{-1}\mathbf{G}^T\mathbf{Z}_{i+1}\right) - \sum_{j=0}^{i} \mathbf{Q}_j\boldsymbol{\Delta}_j^+\boldsymbol{\Phi}_{i,j}$$

One only need to precompute the sparse $\mathbf{FAG}$ (neighbors of neighbors), and to compute the sparse $\mathbf{FZ}_{i+1}$ (neighbors).

Brief analysis of S-FETI

- It can be viewed as a multipreconditioned CG algorithm (Bridson et al., 2006).
- At each iteration it does better than a classical CG would.
- The short recurrence is broken (full reorthogonalization is mandatory)
- Similar idea as Geneo: let the algorithm combine local contributions optimally. But it is done in the CG instead of in the augmentation.

Visualization of S-BDD

Figure: Decomposition with irregular domains

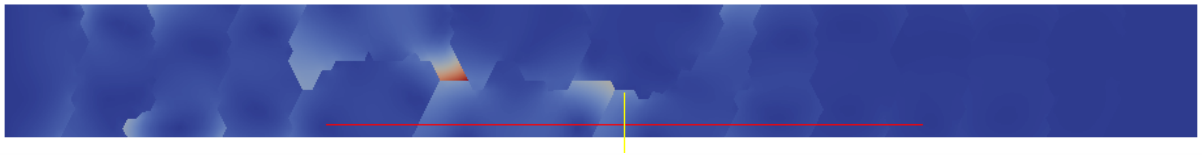

Figure: Local contributions to Z

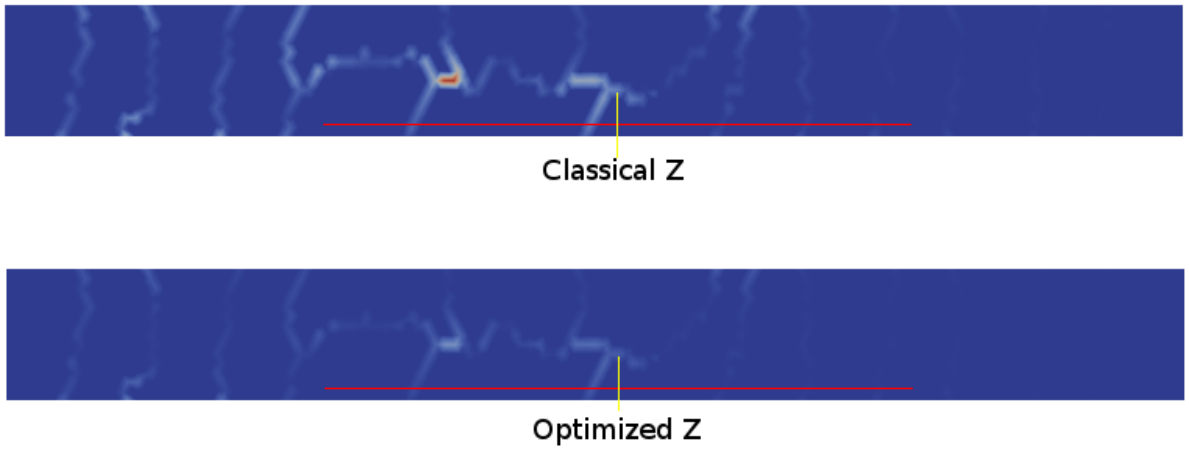

Figure: Classical z vs SFETI z

The irregularity triggers unnecessary local effects which are wiped out by the S-FETI procedure.

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Block-FETI

- S-FETI's main problem is the breaking of the short recurrence.
- Block-FETI is a classical block conjugate gradient (O'Leary, 1980) where a block of right-hand sides is generated in order to activate the local effects: for an initial guess $\boldsymbol{\lambda}_{00}$, the initial residual for the classical and for the block FETI write

$$\mathbf{r}_0 = \mathbf{P}^T \left(\sum_s \mathbf{B}^{(s)} \mathbf{K}^{(s)^+} \left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)^T} (\boldsymbol{\lambda}_0 + \mathbf{P}\boldsymbol{\lambda}_{00}) \right) \right)$$

$$\mathbf{R}_0 = \mathbf{P}^T \left(\ldots, \mathbf{B}^{(s)} \mathbf{K}^{(s)^+} \left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)^T} (\boldsymbol{\lambda}_0 + \mathbf{P}\boldsymbol{\lambda}_{00}) \right), \ldots \right).$$

- We solve $\mathbf{P}^T \mathbf{F} \mathbf{P} \tilde{\boldsymbol{\Lambda}} = \mathbf{R}_0$ with block-CG
- Connection $\tilde{\boldsymbol{\lambda}} = \tilde{\boldsymbol{\Lambda}} \mathbf{1}$ and $\mathbf{r}_0 = \mathbf{R}_0 \mathbf{1}$ (with $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^N$) we can monitor the convergence of the original system.

Algorithm 4: Block FETI full ortho

$$\mathbf{R}_0 = \mathbf{P}^T\left(\ldots, \mathbf{B}^{(s)}(\mathbf{d}^{(s)} - \mathbf{F}^{(s)}\mathbf{B}^{(s)\,T}(\boldsymbol{\lambda}_0 + \mathbf{P}\boldsymbol{\lambda}_{00})), \ldots\right)$$

$\mathbf{Z}_0 = \tilde{\mathbf{S}}\mathbf{R}_0$, $\mathbf{W}_0 = \mathbf{P}\mathbf{Z}_0$, $\tilde{\boldsymbol{\Lambda}}_0 = 0$, $i = 0$

while $\sqrt{\mathbf{1}^T\mathbf{R}_i^T\mathbf{Z}_i\mathbf{1}} > \epsilon$ **do**

- $\mathbf{Q}_i = \mathbf{F}\mathbf{W}_i$
- $\boldsymbol{\Delta}_i = \mathbf{Q}_i^T\mathbf{W}_i$
- $\Gamma_i = \mathbf{R}_i^T\mathbf{Z}_i$
- $\tilde{\boldsymbol{\Lambda}}_{i+1} = \tilde{\boldsymbol{\Lambda}}_i + \mathbf{W}_i\boldsymbol{\Delta}_i^+\Gamma_i$
- $\mathbf{R}_{i+1} = \mathbf{R}_i - \mathbf{P}^T\mathbf{Q}_i\boldsymbol{\Delta}_i^+\Gamma_i$
- $\mathbf{Z}_{i+1} = \tilde{\mathbf{S}}\mathbf{R}_{i+1}$
- $\mathbf{W}_{i+1} = \mathbf{P}\mathbf{Z}_{i+1}$
- for $0 \leqslant j \leqslant i$
  - $\boldsymbol{\Phi}_{i,j} = \mathbf{Q}_j^T\mathbf{W}_{i+1}$
  - $\mathbf{W}_{i+1} \leftarrow \mathbf{W}_{i+1} - \mathbf{W}_j\boldsymbol{\Delta}_j^+\boldsymbol{\Phi}_{i,j}$
- $i \leftarrow i + 1$

end

$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \mathbf{P}\boldsymbol{\lambda}_{00} + \tilde{\boldsymbol{\Lambda}}\mathbf{1}$

Algorithm 5: FETI2 full ortho

$\mathbf{r}_0 = \mathbf{P}^T{\mathbf{P}_\mathbf{C}}^T(\mathbf{d} - \mathbf{F}\boldsymbol{\lambda}_0)$

$\mathbf{z}_0 = \tilde{\mathbf{S}}\mathbf{r}$, $\mathbf{w}_0 = \mathbf{P}\mathbf{z}_0$

$\hat{\boldsymbol{\lambda}}_0 = 0$, $i = 0$

while $\sqrt{\mathbf{r}_i^T\mathbf{z}_i} > \epsilon$ **do**

- $\mathbf{q}_i = {\mathbf{P}_\mathbf{C}}^T\mathbf{F}\mathbf{w}_i$
- $\delta_i = \mathbf{q}_i^T\mathbf{w}_i$
- $\gamma_i = \mathbf{r}_i^T\mathbf{z}_i$
- $\hat{\boldsymbol{\lambda}}_{i+1} = \hat{\boldsymbol{\lambda}}_i + (\gamma_i/\delta_i)\mathbf{w}_i$
- $\mathbf{r}_{i+1} = \mathbf{r}_i - (\gamma_i/\delta_i)\mathbf{P}^T\mathbf{q}_i$
- $\mathbf{z}_{i+1} = \tilde{\mathbf{S}}\mathbf{r}_{i+1}$
- $\mathbf{w}_{i+1} = \mathbf{P}\mathbf{z}_{i+1}$
- for $0 \leqslant j \leqslant i$
  - $\phi_{i,j} = \mathbf{q}_j^T\mathbf{w}_{i+1}$
  - $\mathbf{w}_{i+1} \leftarrow \mathbf{w}_{i+1} - (\phi_{i,j}/\delta_j)\mathbf{w}_j$
- $i \leftarrow i + 1$

end

$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \tilde{\boldsymbol{\lambda}}_0 + \mathbf{P}_\mathbf{C}\hat{\boldsymbol{\lambda}}_i$

Analysis

Size of blocks

$$\boldsymbol{\lambda},\ \boldsymbol{\lambda}_0,\ \boldsymbol{\lambda}_{00}\ \in \mathbb{R}^n; \quad \tilde{\boldsymbol{\Lambda}}_i, \mathbf{Z}_i,\ \mathbf{W}_i,\ \mathbf{Q}_i\ \in \mathbb{R}^{n\times N}; \quad \boldsymbol{\Gamma}_i,\ \boldsymbol{\Delta}_i,\ \boldsymbol{\Phi}_{i,j}\ \in \mathbb{R}^{N\times N};$$

Costs

- Same costs as S-FETI except that the Dirichlet and Neumann problems are solved on N-blocks.
- Risk of linear dependencies between the residuals and need of pseudo-inversion of $\boldsymbol{\Delta}_i$.Need to *deflate* the residual (Nikishin et al., 1995).

Properties

- Classical convergence properties of block CG
- Need of a small random initialization $\boldsymbol{\lambda}_{00}$ to have all right hand sides independent and to activate bad eigenvectors. Similar to bootstrap adaptive multigrid algorithm (Brandt et al., 2011).
- Block of eigenvalues due to local contributions detected at each iteration.

Outlook

1. Motivation and objectives
2. Brief review on FETI
3. Simultaneous FETI
   - The S-FETI Algorithm
   - Cost of S-FETI
   - Brief analysis of S-FETI
4. Block-FETI
   - The block FETI algorithm
   - Brief analysis
5. Assessments
   - High heterogeneity
   - Bad aspect-ratio
   - Irregular interfaces
   - Decomposition with multiple points
   - Incompressibility
   - Bigger problems
6. Conclusion

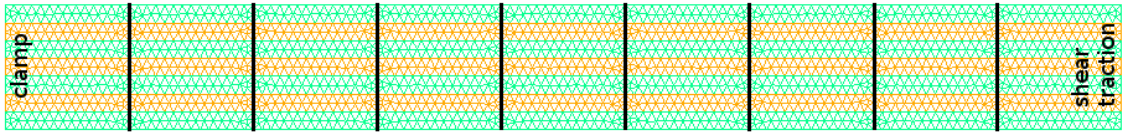


Figure: Heterogeneous beam, 3 stiff (orange) fibers are imbedded in a soft (green) material, 9-subdomain band decomposition.

$\frac{E_{stiff}}{E_{soft}}$	1	10	100	10^3	10^4	10^5	10^6
# iterations FETI $\mathbf{P_I}$	6	8	16	29	44	57	63
# iterations FETI $\mathbf{P_{\tilde{S}}}$	6	6	9	18	31	41	43
# iterations S-FETI $\mathbf{P_I}$	5	6	8	10	11	10	10
# iterations S-FETI $\mathbf{P_{\tilde{S}}}$	5	6	8	9	9	9	8
# iterations B-FETI$\mathbf{P_I}$	5	6	7	8	9	9	9
# iterations B-FETI $\mathbf{P_{\tilde{S}}}$	5	6	6	10	12	11	11

Table: Number of FETI iteration to decrease the initial residual by a 10^6 factor depending on the level of heterogeneity

Same example as previous, homogeneous material, we do a dilatation in the y-direction.

Aspect ratio	1/5	1	5	10
# iterations FETI $\mathbf{P_I}$	6	6	24	34
# iterations FETI $\mathbf{P_{\tilde{S}}}$	5	6	24	34
# iterations S-FETI $\mathbf{P_I}$	5	5	9	11
# iterations B-FETI $\mathbf{P_I}$	5	5	8	10

Table: Number of FETI iteration to decrease the initial residual by a 10^6 factor depending on the aspect ratio

The shape of interfaces is known to have a strong influence on the convergence of the solver (Klawonn et al., 2008). Roughly, the straighter the better. The irregular decomposition of the beam was obtained by an automatic graph partitioner.

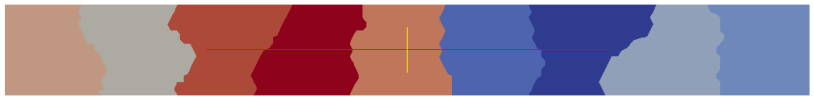
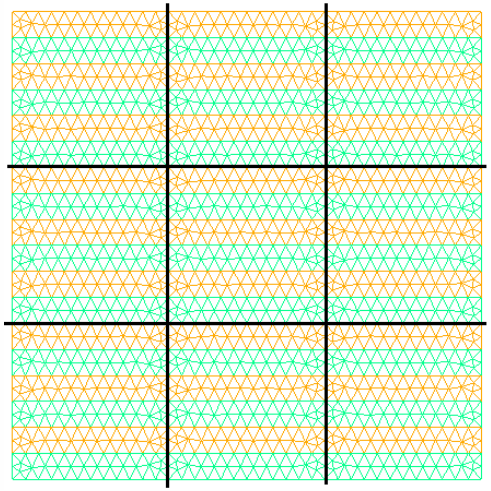


Figure: Homogeneous beam with irregular interfaces

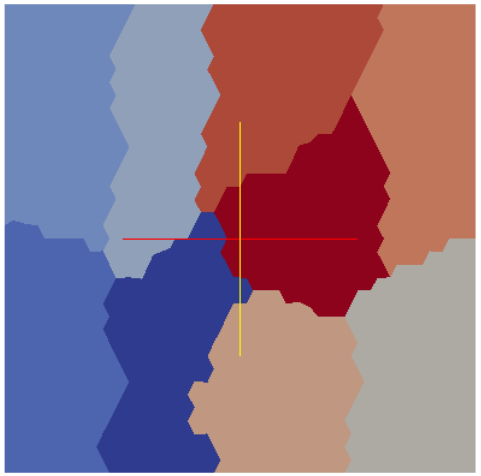
Decomposition	straight	irregular
# iterations FETI $\mathbf{P_I}$	6	12
# iterations FETI $\mathbf{P_{\tilde{S}}}$	6	11
# iterations S-FETI $\mathbf{P_I}$	5	8
# iterations B-FETI $\mathbf{P_I}$	5	8

Table: Number of FETI iteration to decrease the initial residual by a 10^6 factor depending on the decomposition

Multiple points are not truly speaking the cause of difficulties, in particular since (D. Rixen et al., 1999) in the case of heterogeneous structures and (C. Farhat et al., 1998) in the case of plates and shells problems.



(a) Heterogeneous square regular decomposition



(b) Automatic decomposition

Heterogeneity ratio	1	1	10^5	10^5
Decomposition	straight	irregular	straight	irregular
# iterations FETI $\mathbf{P_I}$	12	17	44	93
# iterations FETI $\mathbf{P_{\tilde{S}}}$	12	19	46	93
# iterations S-FETI $\mathbf{P_I}$	8	8	12	16
# iterations B-FETI $\mathbf{P_I}$	7	7	9	11

Table: Number of FETI iterations to decrease the initial residual by a 10^6 factor.

Incompressibility is also a known factor for convergence difficulties, a classical cure is to add a coarse problem related to the conservation of the volume of the subdomains (Vereecke et al., 2003).
We consider the geometry of the beam with homogeneous linear elastic material in plane strain. The bottom and top faces are clamped and a pressure is imposed on the left side whereas the right side is free. Table 5 gives the number of iterations to converge for various Poisson coefficients close to the incompressible limit $\nu \simeq 0.5$.

	$1/2 - \nu = 10^{-1}$	$1/2 - \nu = 10^{-5}$	$1/2 - \nu = 10^{-6}$
# iterations FETI	5	31	63
# iterations S-FETI	5	18	23
# iterations B-FETI	5	18	22

Table: Number of FETI iterations to decrease the initial residual by a 10^6 factor for the quasi-incompressible problem.

First results in larger FE code

Implementation in progress in Z-Set by A. Parret-Fréaud and C. Bovet

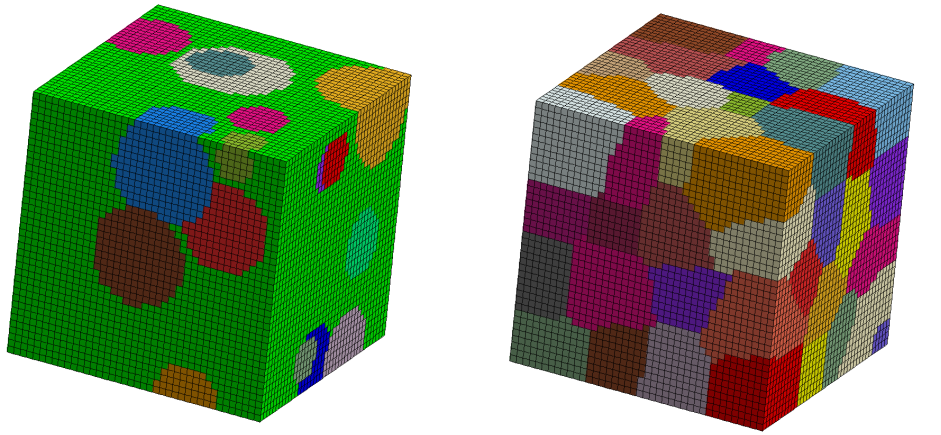


Figure: Cube, 400 000 dof, 10^6 elasticity contrast, jagged interfaces

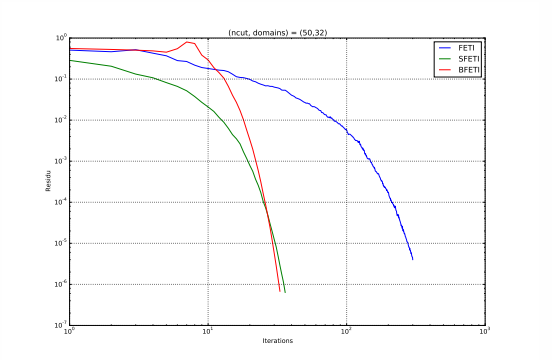


Figure: Typical convergence

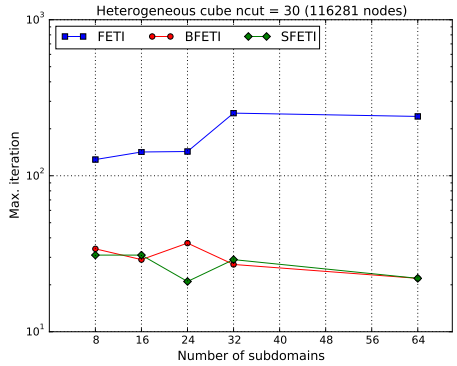


Figure:

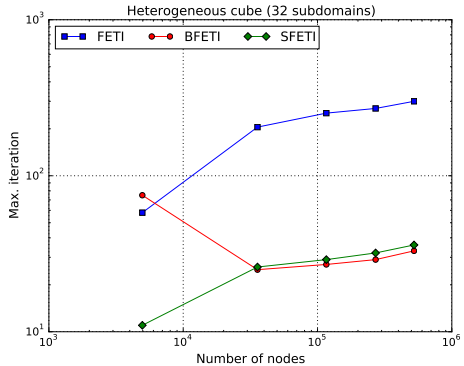


Figure:

Conclusion

- We presented two block-based strategies for FETI: Simultaneous-FETI (multipreconditioned CG) and Block-FETI (Block-CG with RHS made out of local contributions).
- The aim is to do like Geneo without the prior analysis: to detect local contributions which are not well approximated by the classical preconditioner.
- Extra operations are very limited, efficiently implemented (blocks) and adapted to parallelism (large exchanges but as frequent).
- Performance is very encouraging: stability of the solver with respect to heterogeneity, bad interfaces, bad aspect ratio, incompressibility.
- We expect scalability problems for large number of subdomains, investigation in progress ...

A paper is online in IJNME, (preprint on hal).

Wait for Nicole's presentation in DD23 for improved algorithms.

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