A dual algorithm for denoising and preserving edges in image processing

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Image restoration

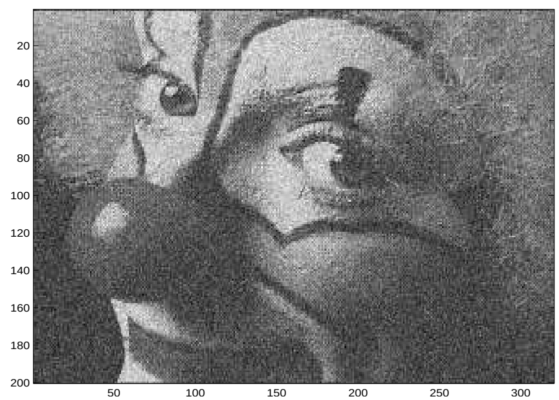
- Noisy image :

$$f = u + \varepsilon$$

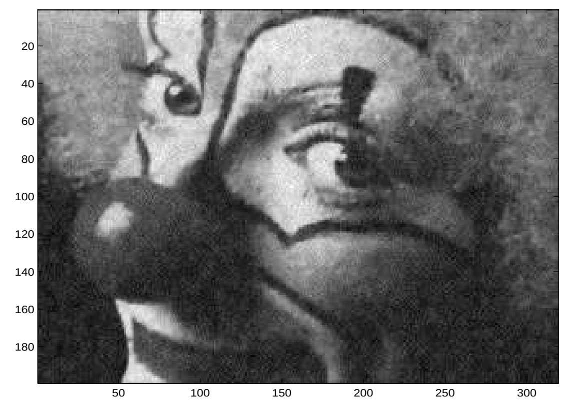
with : ε = noise, not a smooth function ; $\varepsilon \in L^2(\Omega)$ or $L^\infty(\Omega)$

Goal : Recover from a blurred image a smooth one, as close as possible to the original. Needs to :

  - Rub out small noises
  - Restore original discontinuities that have been erased by noises



Noisy image



Processed image

Image restoration

- Inverse problem : Recovering data from the observed image

$\Longrightarrow$ least squares fitting

$$\begin{cases} \text{Find } u \in V \text{ (a proper space) such that :} \\ \\ ||u - f||^2 = \min_{v \in V} ||v - f||^2_{L^2(\Omega)} \end{cases}$$

- Difficulties :

  - Usually, this minimum is zero
  - No element in V realizes it (lack of coercivity)

$\Longrightarrow$ Regularization is needed to restore nice properties.

A smoothed TV approach

(G. Aubert, P. Kornprobst, 2001)

$$\min_{v \in BV(\Omega)} \left\{ J(v) = \frac{1}{2} \int_\Omega |f - v|^2 + \varepsilon \int_\Omega \sqrt{1 + a\,|\nabla v|^2} \right\}$$

- For small $|\nabla v|$ $\quad \sqrt{1 + a\,|\nabla v|^2} \sim 1 + \frac{a}{2}\,|\nabla v|^2$ and thus

$$J(v) \sim \frac{1}{2}\,||v - f||^2_{L^2} + \frac{\varepsilon a}{2}\,||\nabla v||^2_{L^2}$$

Looks like a Tikhonov regularization

- For large $|\nabla v|$ $\quad \sqrt{1 + a\,|\nabla v|^2} \sim \sqrt{a}\,|\nabla v|$ and thus

$$J(v) \sim \frac{1}{2}\,||v - f||^2_{L^2} + \varepsilon\sqrt{a}\,||\nabla v||_{L^1}$$

Looks like a Total Variation regularization

A direct approach

(P. Destuynder, O. Wilk, 2004)

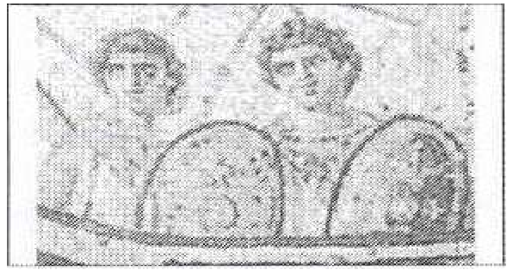
Optimality condition yields :

$$(\mathcal{O}) \qquad \varepsilon\, a \int_\Omega \frac{\nabla u . \nabla v}{\sqrt{1 + a\, |\nabla u|^2}} + \int_\Omega u\, v = \int_\Omega f v \qquad \forall v \in H^1(\Omega)$$

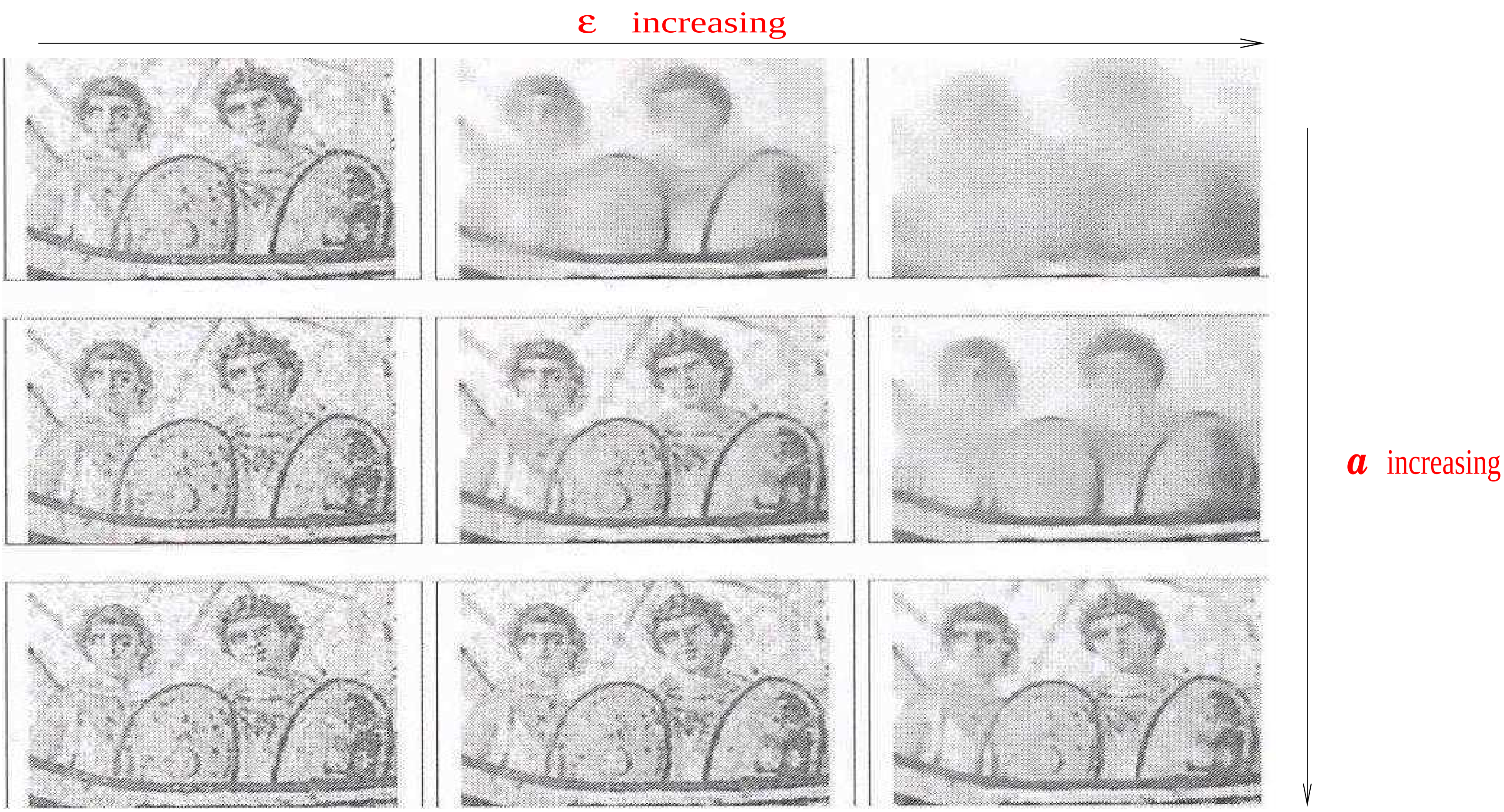
Suppose $u \in W^{1,\infty}(\Omega)$ and use a fixed point algorithm to solve $(\mathcal{O})$

$$\begin{cases} \text{Find } u^{p+1} \in W^{1,\infty}(\Omega) \text{ such that } \forall v \in H^1(\Omega) : \\ \varepsilon\, a \int_\Omega \frac{\nabla u^{p+1} . \nabla v}{\sqrt{1 + a\, |\nabla u^p|^2}} + \int_\Omega u^{p+1} v = \int_\Omega f v \end{cases}$$

A direct approach



Original image



Results of restoration

The dual approach

Set $q = \nabla v$ and suppose $q \in (L^2(\Omega))^2$. The restoration problem becomes :

$$(\mathcal{R}) \begin{cases} \min\limits_{(v,q)\in BV(\Omega)\times(L^2(\Omega))^2} \dfrac{1}{2}\displaystyle\int_\Omega |f-v|^2 + \varepsilon \int_\Omega \sqrt{1 + a\ |q|^2} \\ q = \nabla v \end{cases}$$

Dualize the constraint and get :

$$\mathcal{L}(v,q,\mu) = \frac{1}{2}\int_\Omega |f-v|^2 + \varepsilon \int_\Omega \sqrt{1 + a\ |q|^2} + \int_\Omega (\nabla v - q)\mu$$

$(v,q) \in BV(\Omega) \times (L^2(\Omega))^2$

$\mu \in H_0(\mathrm{div},\Omega) = \{\mu \in (L^2(\Omega))^2, \mathrm{div}\mu \in L^2(\Omega),\ \mu.\nu = 0 \text{ on } \partial\Omega\}$

The dual approach

The restoration problem $(\mathcal{R})$ is thus equivalent to :

$$(\mathcal{P}) \qquad \min_{(v,q)\in BV(\Omega)\times(L^2(\Omega))^2} \; \sup_{\mu\in H_0(\text{div},\Omega)} \mathcal{L}(v,q,\mu)$$

the dual problem of which is :

$$(\mathcal{P}^*) \qquad \max_{\mu\in H_0(\text{div},\Omega)} \; \min_{(v,q)\in BV(\Omega)\times(L^2(\Omega))^2} \mathcal{L}(v,q,\mu)$$

We have :

- $\max\limits_{\mu} \min\limits_{(v,q)} \mathcal{L}(v,q,\mu) \quad \leq \quad \min\limits_{(v,q)} \max\limits_{\mu} \mathcal{L}(v,q,\mu)$

- Problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ are not equivalent (duality gap)...unless additional regularity is available

- However solving the dual problem may bring useful informations

Solving the dual problem

1_ Given μ, solve the minimum problem with respect to (v, q)

$$\mathcal{L}(u(\mu), p(\mu), \mu) = \min_{(v,q)\in BV(\Omega)\times(L^2(\Omega))^2} \mathcal{L}(v, q, \mu)$$

Optimality $\Longrightarrow$

- $\dfrac{\partial \mathcal{L}}{\partial v} = 0 \Rightarrow u(\mu) = f + \mathrm{div}\mu$

- $\dfrac{\partial \mathcal{L}}{\partial q} = 0 \Rightarrow p(\mu) = \dfrac{\mu}{\sqrt{a}\sqrt{\varepsilon^2 a - |\mu|^2}}$

Solving the dual problem

2_ Replace $u(\mu)$ and $p(\mu)$ and get :

$$G(\mu) = -\mathcal{L}(u, p, \mu) = \frac{1}{2}\int_\Omega |\text{div}\mu|^2 + \int_\Omega f\,\text{div}\mu - \frac{1}{\sqrt{a}}\int_\Omega \sqrt{a\,\varepsilon^2 - |\mu|^2}$$

$$\mu \in K = \{\mu \,/\, \mu \in H_0(\text{div}, \Omega), |\mu| \leq \varepsilon\,\sqrt{a}\}$$

3_ Solve now the optimization problem with respect to the multiplier μ

$$G(\lambda) = \min_{\mu \in K} G(\mu)$$

4_ Replace λ and get final solution

$$\begin{cases} u = f + \text{div}\lambda \\ p = \dfrac{\lambda}{\sqrt{a}\sqrt{\varepsilon^2 a - |\lambda|^2}} \end{cases}$$

Solving the dual problem

Change variables to get a nicer problem :

Set $\eta = \dfrac{\mu}{\sqrt{a}}$ and $z = \sqrt{\varepsilon^2 - \eta^2}$, then $\eta \in K_\varepsilon = \{\eta \in H_0(\text{div}, \Omega); |\eta| \leq \varepsilon\}$.

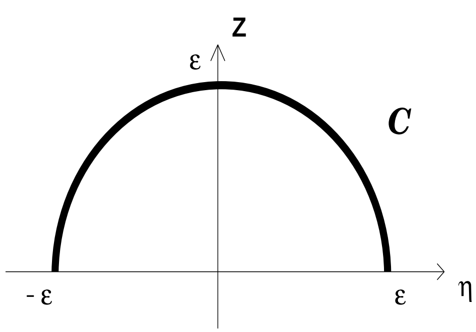
Then

$$\min_{(\eta,z)\in\mathcal{C}} \frac{a}{2} \int_\Omega |\text{div }\eta|^2 + \sqrt{a} \int_\Omega f \text{ div }\eta - \int_\Omega z$$

$\mathcal{C} = \{(\eta, z) \in K_\varepsilon \times L^2(\Omega) \quad / \quad \eta^2 + z^2 = \varepsilon^2; z \geq 0\}$

Two difficulties to go through

1_ The constraint is not convex



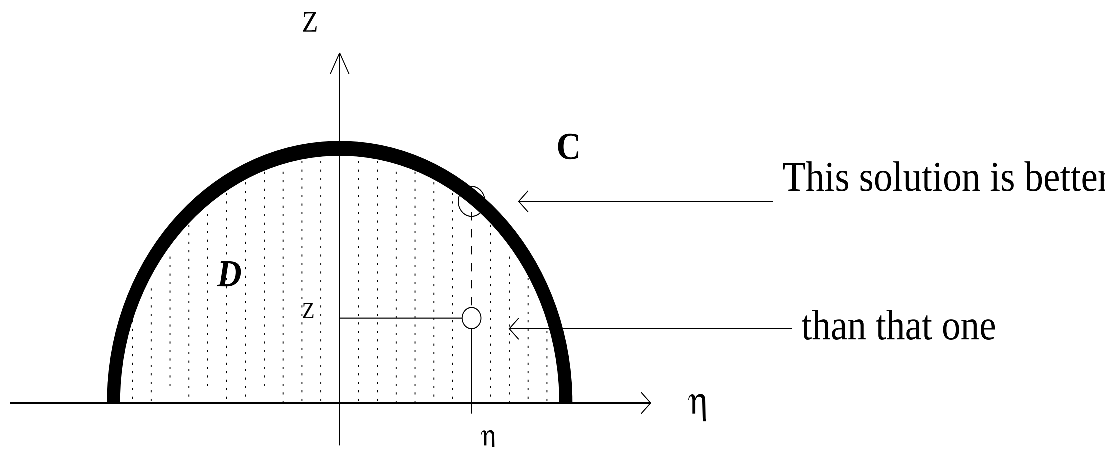
2_ Uniqueness of η is only up to a divergence free function

Solving the dual problem

1_ Convexity

Replace the non convex set $\mathcal{C}$ by the convex one $\mathcal{D}$

$$\mathcal{D} = \left\{(\eta, z) \in H_0(\mathrm{div}, \Omega) \times L^2(\Omega) \quad / \quad \eta^2 + z^2 \leq \varepsilon \quad ; \quad z \geq 0\right\}$$



Minimizing on $\mathcal{D}$ is actually equivalent to minimizing on $\mathcal{C}$

$$\min_{(\eta,z)\in\mathcal{D}} \frac{a}{2}\int_\Omega |\mathrm{div}\ \eta|^2 + \sqrt{a}\int_\Omega f \mathrm{div}\ \eta - \int_\Omega z$$

Solving the dual problem

2_ Coercivity

Adding a regularization term restores coercivity

$$\min_{(\eta,z)\in\mathcal{D}} \frac{a}{2}\int_\Omega |\mathrm{div}\ \eta|^2 + \sqrt{a}\int_\Omega f\,\mathrm{div}\ \eta + \frac{a\beta}{2}\int_\Omega |\eta|^2 - \int_\Omega z$$

$\Longrightarrow$ Existence and uniqueness of a solution $(\eta^\beta, z^\beta) \in \mathcal{C}$

Solving the dual problem

Behaviour of η^β when $\beta \to 0$

- η^β is bounded in $[L^\infty(\Omega)]^2 \Longrightarrow$ bounded in $[L^2(\Omega)]^2$
- There exists a subset $\eta^\beta \in [L^2(\Omega)]^2$ such that :

$$\eta^\beta \rightharpoonup \eta^* \qquad (\text{actually, the one with minimum } L^2 \text{ norm})$$

$\Longrightarrow$ The convergence is weak, but we have :

$$\text{div } \eta^\beta \xrightarrow{L^2(\Omega)} \text{div } \eta^* \text{ when } \beta \to 0$$

The processed image is $u = f + \sqrt{a} \text{ div } \eta^*$ and

$\Longrightarrow$ We are only interested in div η^β

$$u^\beta = f + \sqrt{a} \text{ div } \eta^\beta \xrightarrow{L^2(\Omega)} u = f + \sqrt{a} \text{ div } \eta^* \text{ when } \beta \to 0$$

Solving the dual problem

Gradient algorithm with projection

- First guess (η_0, z_0)

- $k-$th iteration

$$(\eta_{k+1}, z_{k+1}) = \Pi_{\mathcal{D}}((\eta_k, z_k) - \rho_k \nabla \mathcal{H}_\beta(\eta_k, z_k))$$

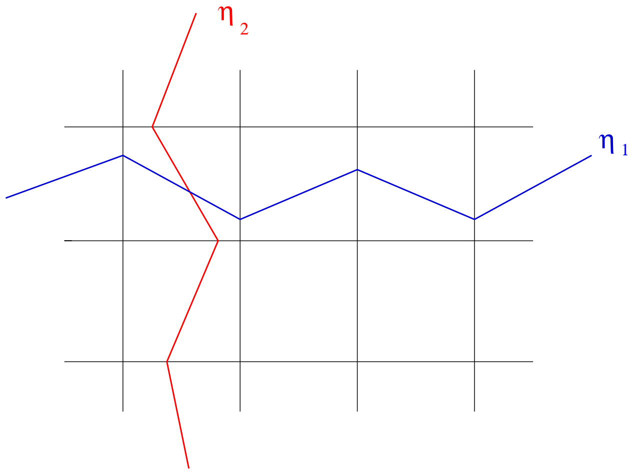
- Stopping criterion : $||\eta_{k+1} - \eta_k||_{(L^\infty)^2} \leq \alpha$ (α is a given threshold)

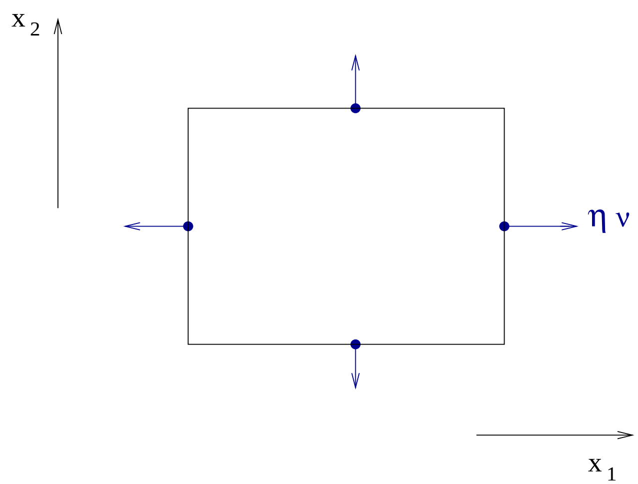
with $\mathcal{H}_\beta(\eta, z) = \dfrac{a}{2}\displaystyle\int_\Omega |\text{div } \eta|^2 + \sqrt{a}\int_\Omega f \text{div } \eta + \frac{a\beta}{2}\int_\Omega |\eta|^2 - \int_\Omega z$

Numerical approximation

Approximation of $H_0(\text{div}, \Omega)$: use the Raviart-Thomas finite elements

$$\eta(x_1, x_2) = \begin{cases} a\, x_1 + b \\ c\, x_2 + d \end{cases}$$



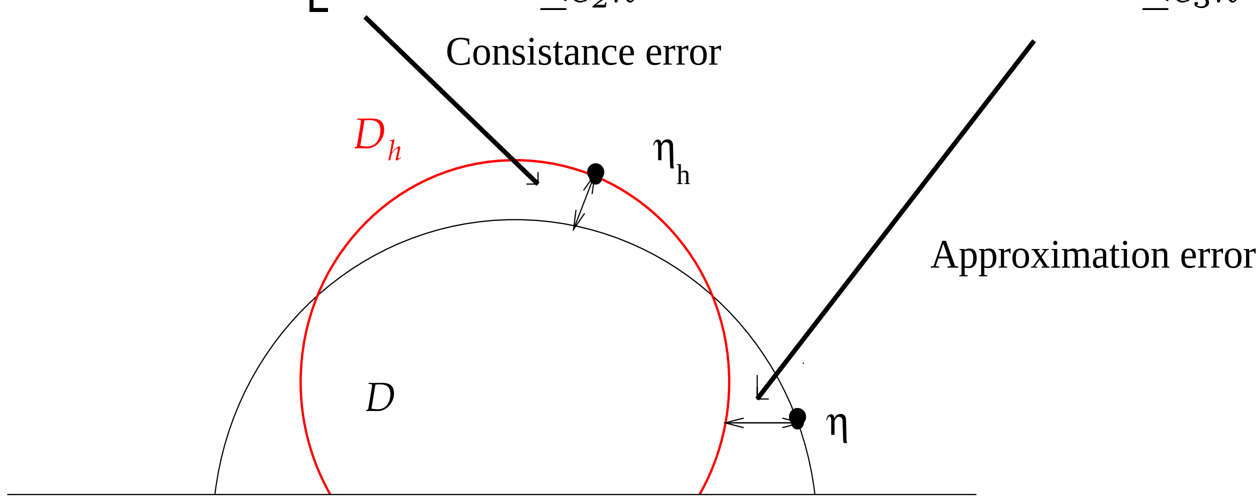


- $\eta.\nu$ constant on cell interfaces, and continuous
- $\eta.\tau$ discontinuous on cell interfaces

Error estimates

By the use of the variational inequation associated to the dual problem, there exists a constant $c_1(f, \beta, \varepsilon, a)$ such that :

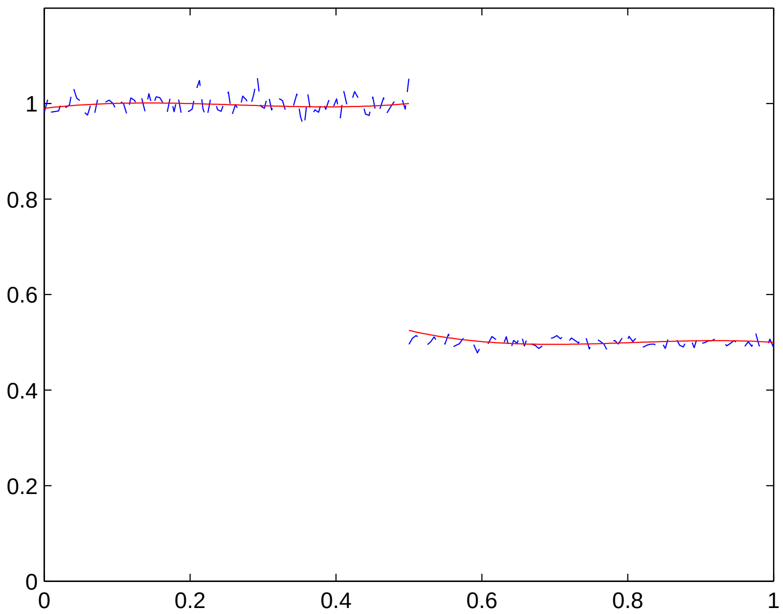
$$\|\eta_h - \eta\|^2_{H(\mathrm{div},\Omega)} \leq c_1 \left[\underbrace{\inf_{\psi \in \mathcal{D}} \|\eta_h - \psi\|_{H(\mathrm{div},\Omega)}}_{\leq c_2 h} + \underbrace{\inf_{\psi_h \in \mathcal{D}_h} \|\eta - \psi_h\|_{H(\mathrm{div},\Omega)}}_{\leq c_3 h} \right]$$

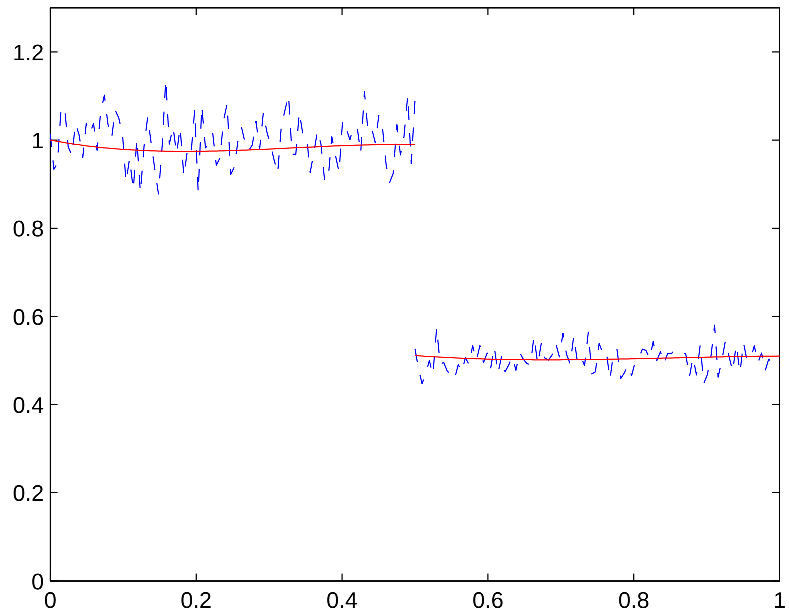


$\Longrightarrow$ We deduce the following estimate

$$\|\eta_h - \eta\|_{H(\mathrm{div},\Omega)} \leq c(f, \beta, \varepsilon, a)\sqrt{h}$$

1D Results

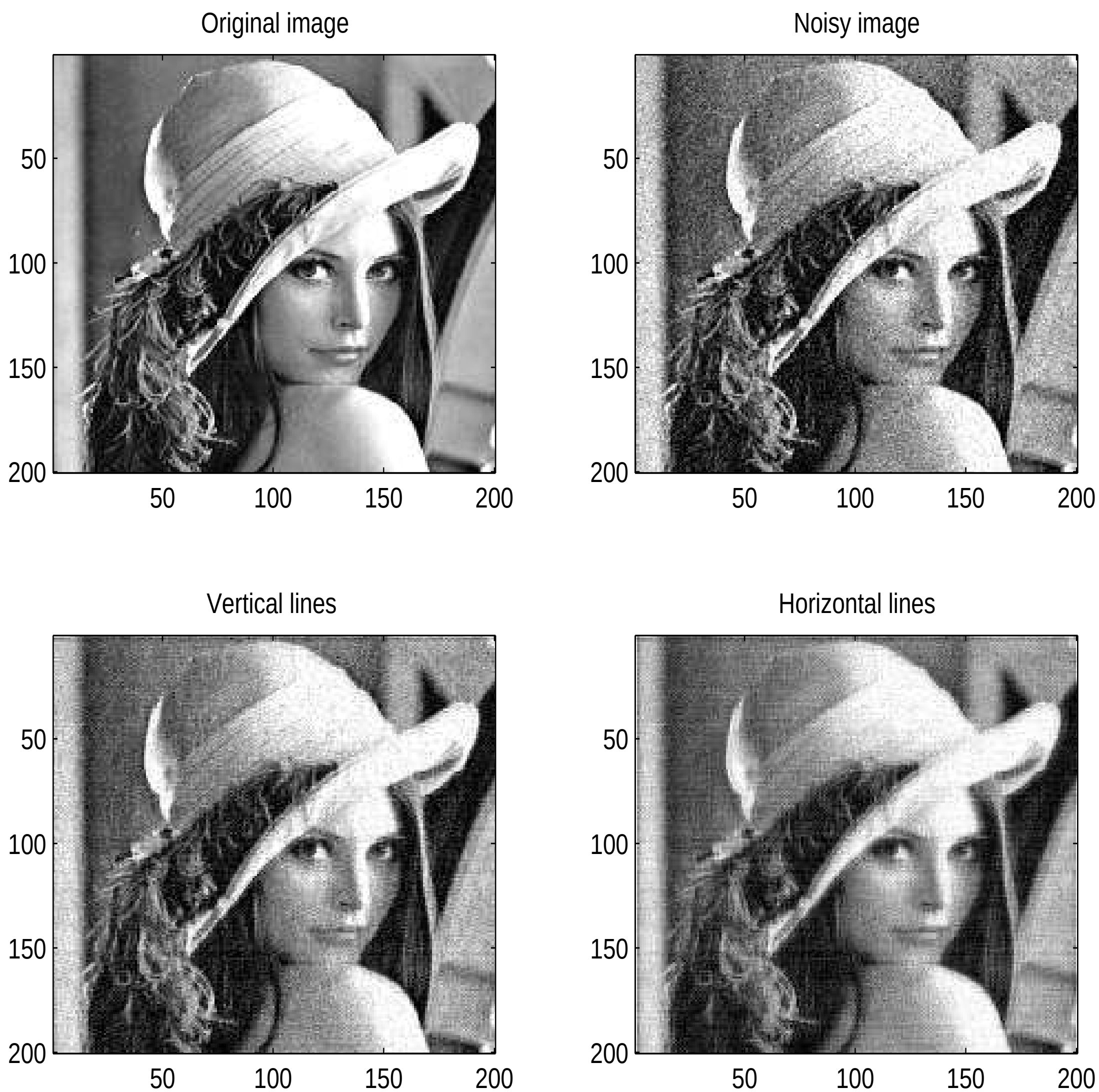




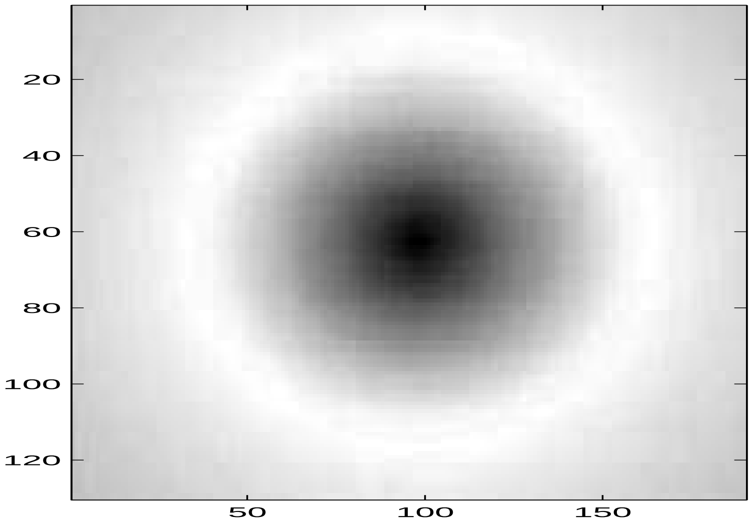
Noise $= 2\%$,$\varepsilon = 10^{-3}, a = 1$

Noise $= 5\%$,$\varepsilon = 10^{-3}, a = 1$

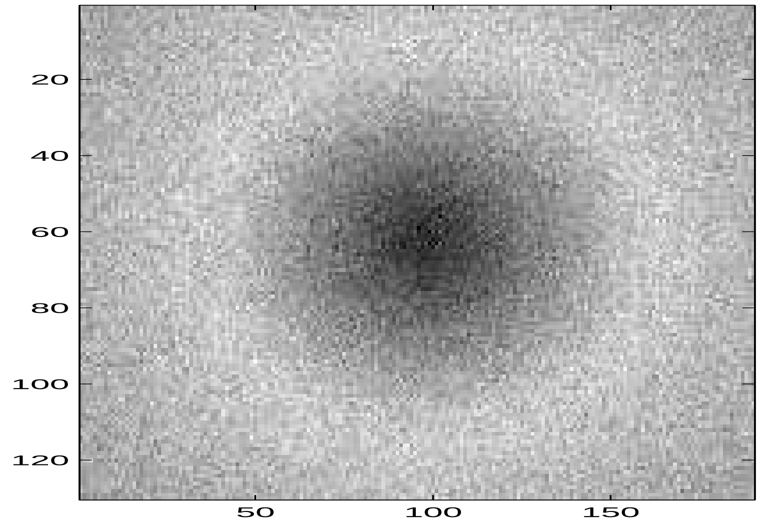
1D-2D Results



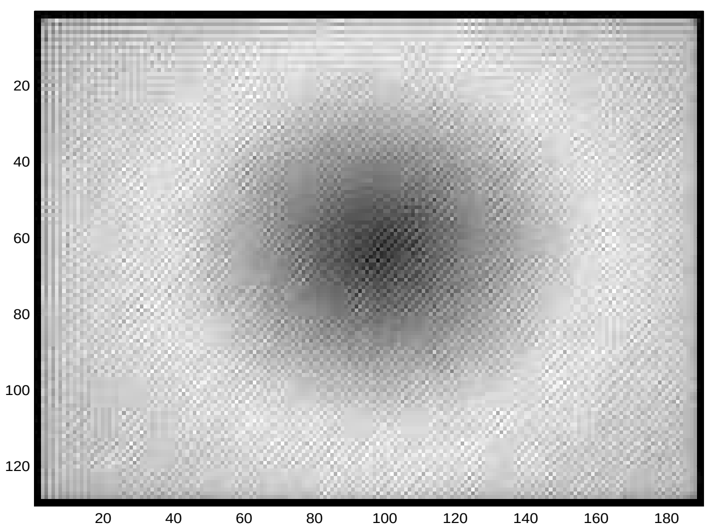
2D Results



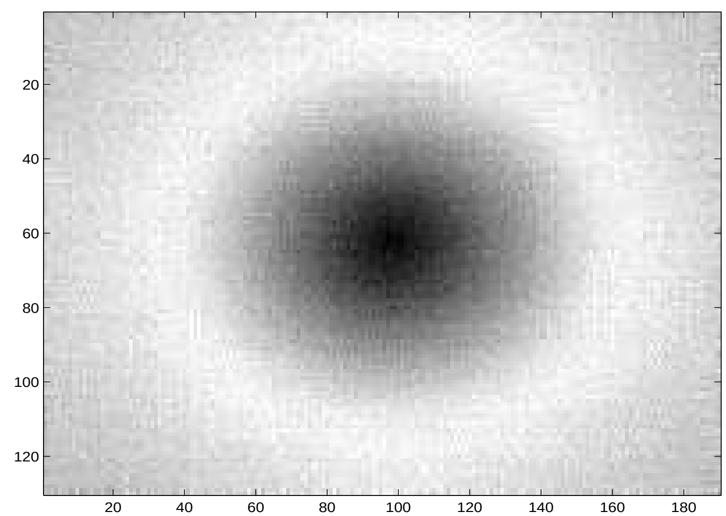
Original image



Noisy image 20%

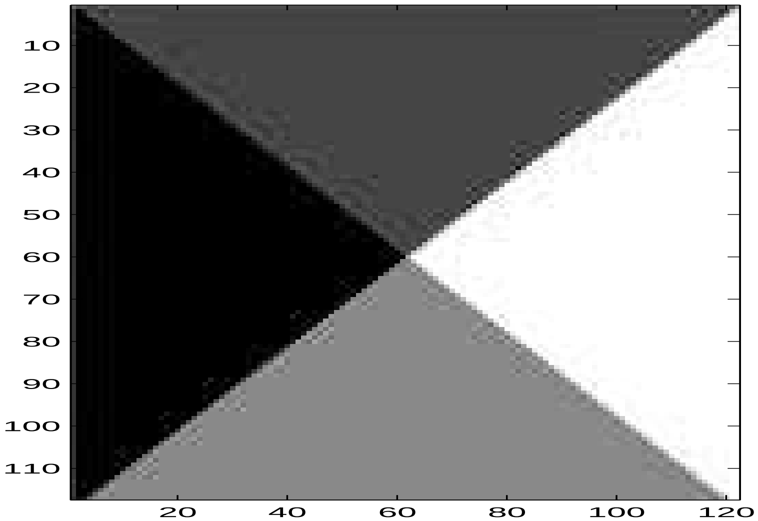


Tikhonov restoration

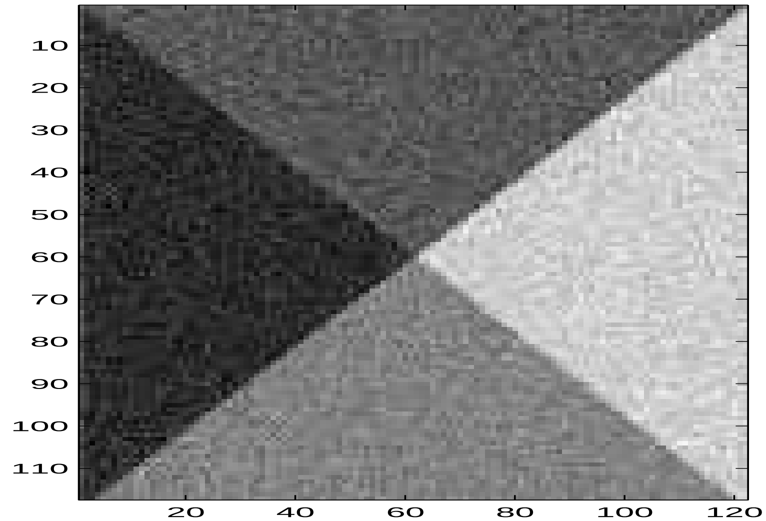


Smoothed TV restoration
$\varepsilon = 10, a = 10^{-2}$

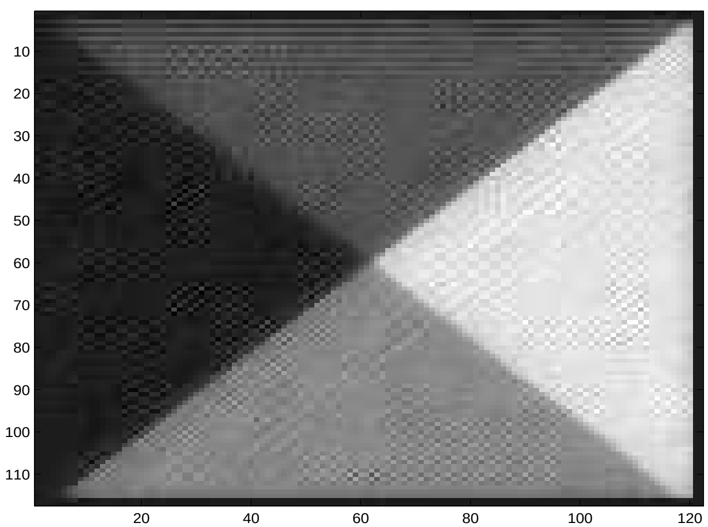
2D Results



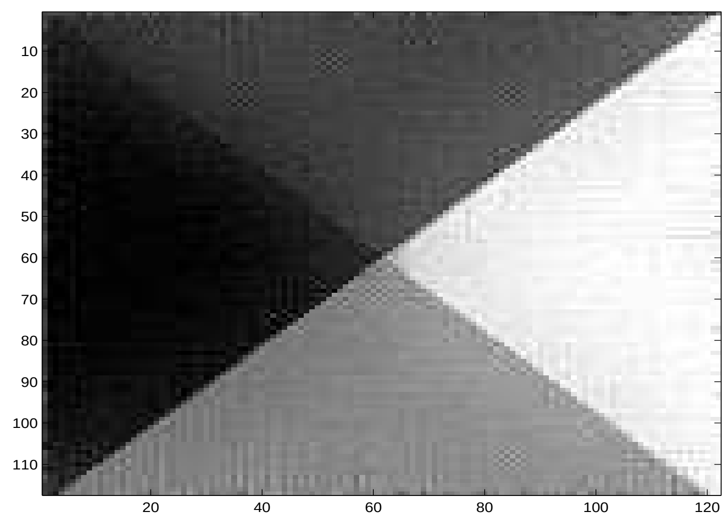
Original image



Noisy image 20%

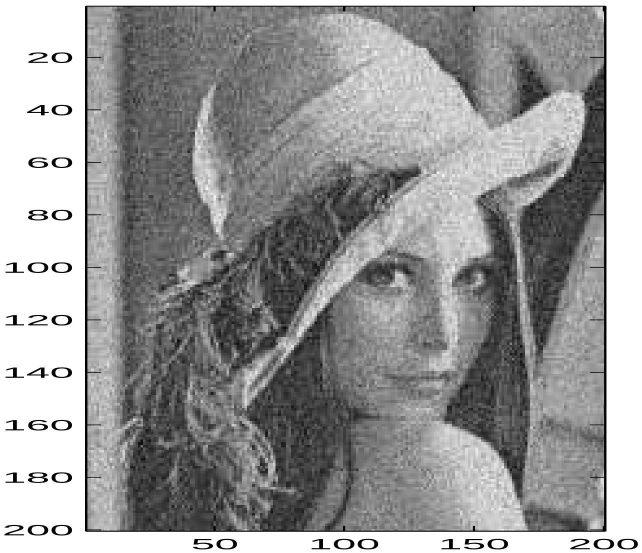


Tikhonov restoration

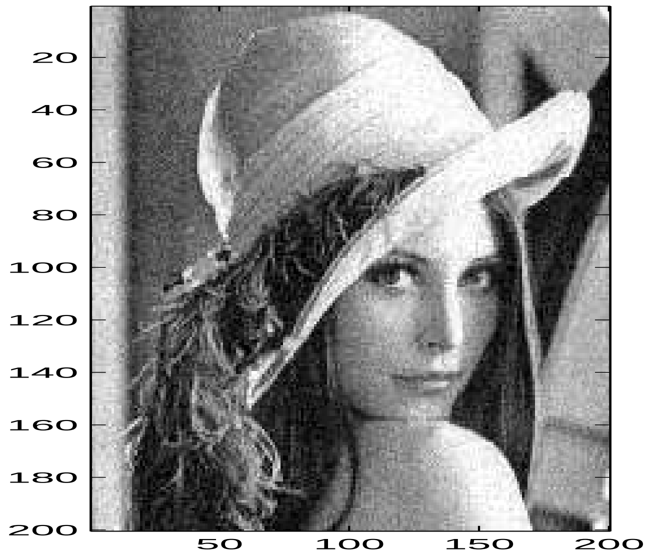


Smoothed TV restoration
$\varepsilon = 10, a = 10^{-2}$

2D Results

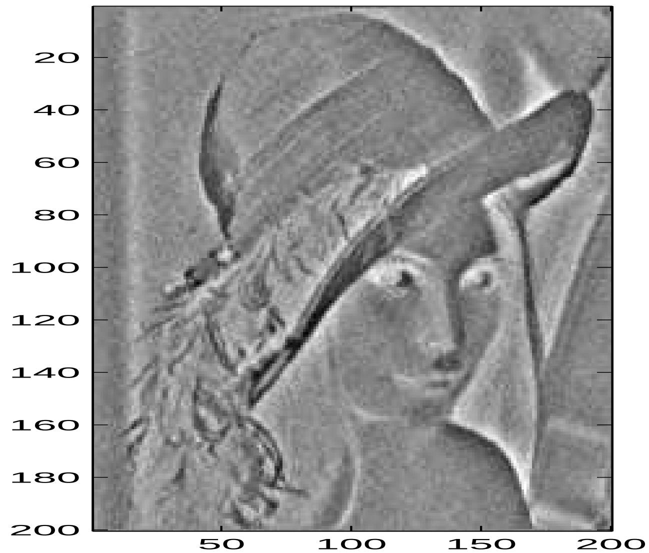


Noisy image



Processed image

$\varepsilon = 10, a = 10^{-2}$

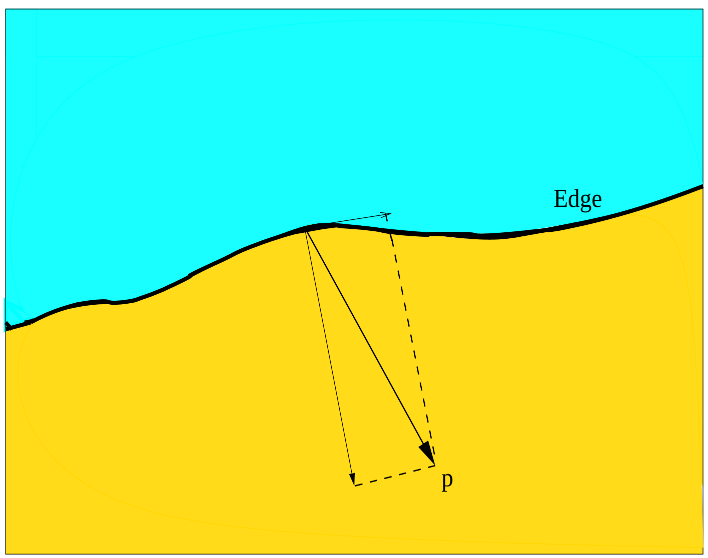


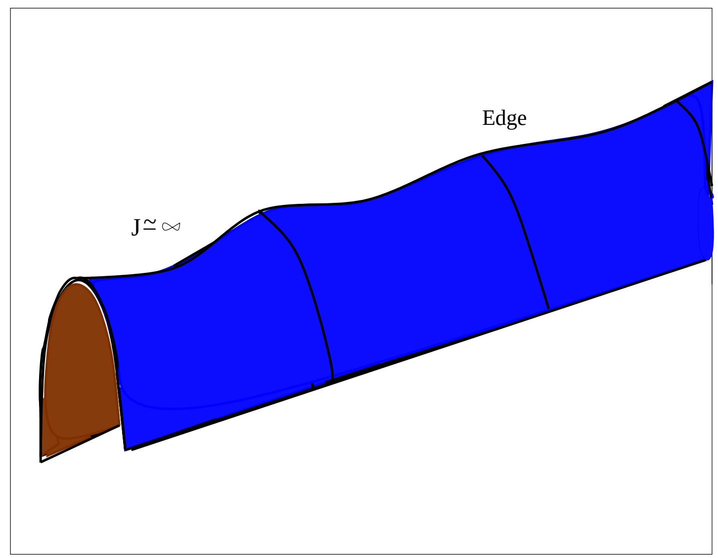
div λ

Contours location

Edges = Large gradient $\qquad \|p\| = \|\nabla u\| = \dfrac{\|\lambda\|}{\sqrt{a}\sqrt{\varepsilon^2 a - |\lambda|^2}}$

$$\left.\begin{array}{l}\text{Large variations across the edge}\\ \text{Small variations along the edge}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\|p\|_\infty \simeq \infty \text{ on the edge}\\ \|p\|_\infty \simeq 0 \text{ on both sides of the edge}\end{array}\right.$$





Contours location

Therefore

$$\text{Edge points} \quad = \quad \begin{cases} \bullet \text{ Almost critical points of } J(x) = |p(x)|^2 \\ \bullet \text{ Eigenvalue } (\nabla^2 J(x)) < 0 \end{cases}$$

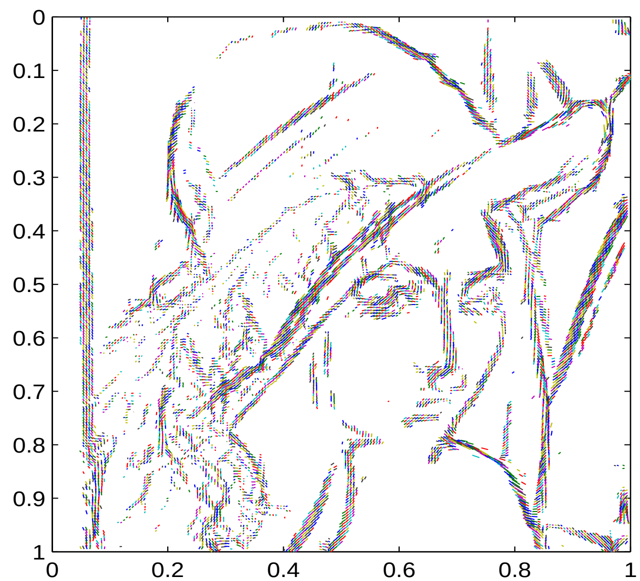
Morever λ is orthogonal to the edge and $|\lambda| \simeq \varepsilon\sqrt{a}$ on the edge (since $|p| \simeq \infty$)

$\Longrightarrow$ Plotting $\lambda^{\perp}$ provides with a small part of the edge

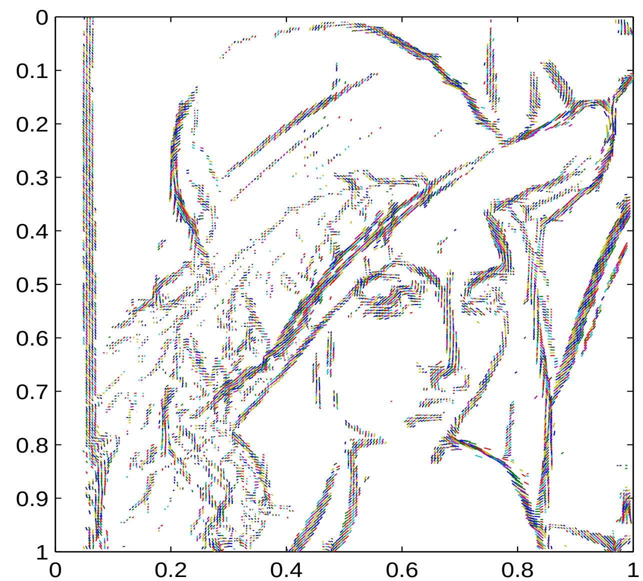
Contours location

Original image

Characterization points



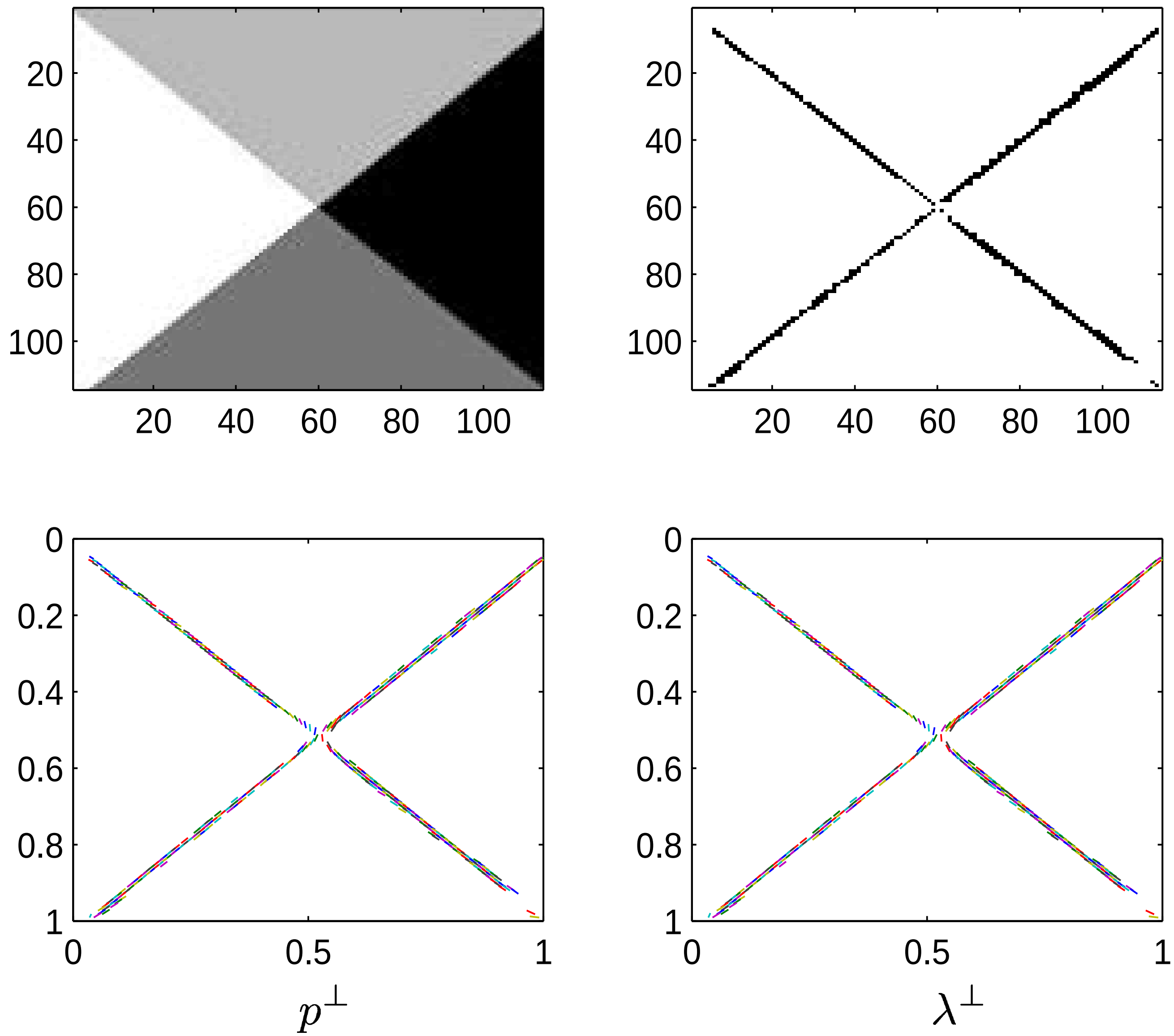
$p^{\perp}$



$\lambda^{\perp}$

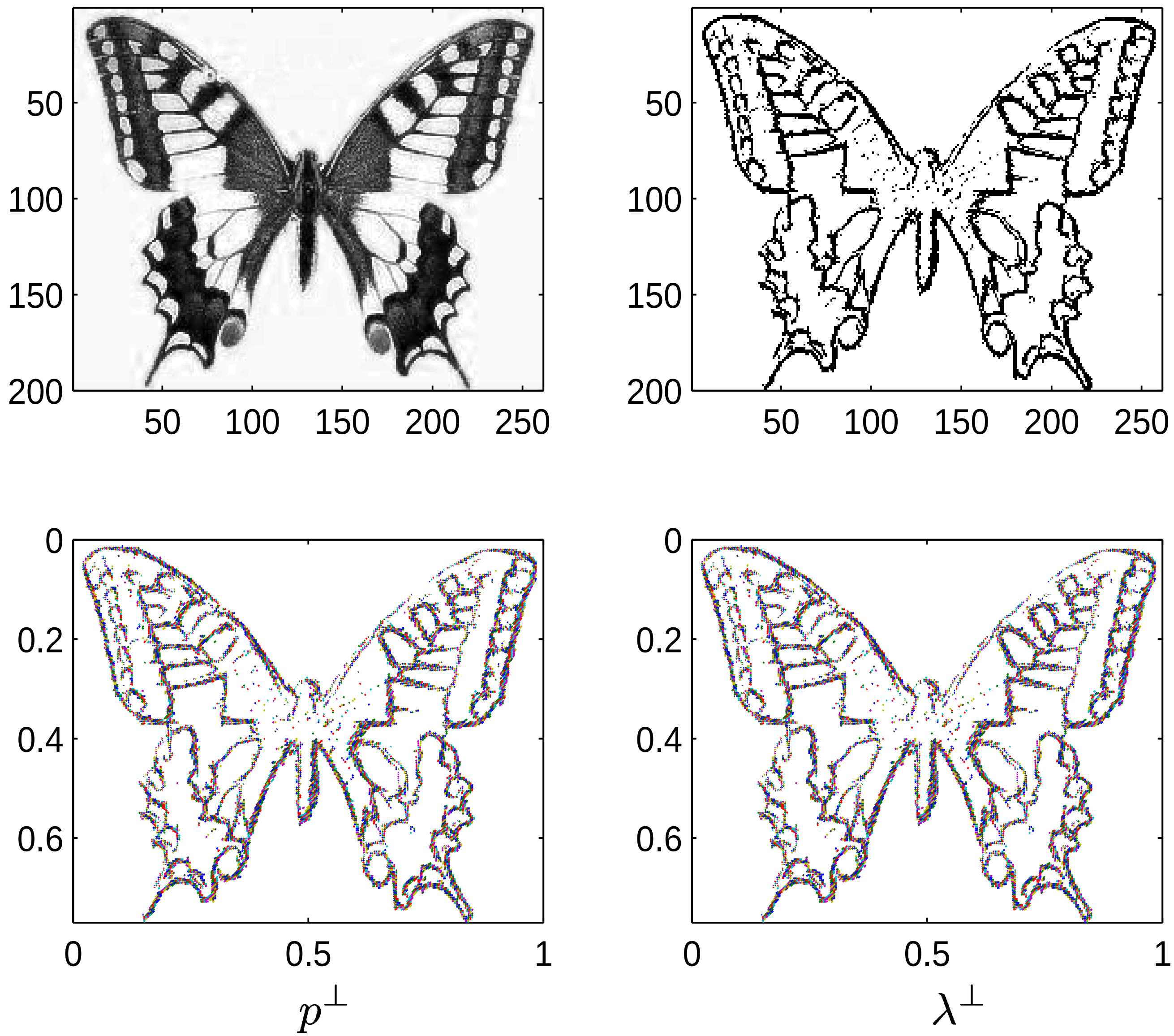
Contours location

Original image | Characterization points



Contours location

Original image　　　　Characterization points



Conclusions

- A dual and fast algorithm to restore images
- Additional gain : Get the image edges from Lagrange multiplier

Still coming up :

- Alternative regularized TV variation ?
- Multiplicative noise ?
- Deblurring images ?