Non-Differentiable Embedding of Lagrangian structures

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Position of the problem

1. Example We consider particles *moving continuously* along a path $x(t)$, of mass 1, under a potential field U. The trajectory is given by the Newton equation

$$\frac{d^2}{dt^2}x(t) = -\nabla U(x).$$

2 Assumption: the trajectories are smooth.

▸ Example

3 Work in Physics from L. Nottale: no hypothesis concerning the differentiability. $\rightarrow$ Natural trajectories are everywhere non-differentiable.
Nottale's idea: take into account this loss of differentiability on the micro-scale.

4 Idea Extension of the notion of derivative.

5 Non-differentiable embedding

- an ODE is the restriction of a more general "differentiable" equation.
  $\longrightarrow$ Non-differentiable embedding of ODE.
- Conservation of the structure of the original ODE by the embedding procedure ?

6 Newton's equation derives from a variational principle associated to a function L called Lagrangian and given here by

$$L(t, x, v) = \frac{1}{2}v^2 - U(x).$$

In fact the trajectories solution of the Newton equation are extremals of the Lagrangian functional $\mathcal{L}$ defined by

$$\mathcal{L}(x) = \int_a^b L(t, x(t), x'(t))dt\,.$$

Lagrangian system is the input of a *Lagrangian* and a variational principle also called *least-action principle*.

Outline

Diagram

1. Classical calculus of variations
2. Non-differentiable embedding
   - Quantum calculus
   - Non-differentiable embedding of Lagrangian systems
   - Non-differentiable calculus of variations
   - Coherence principle
3. Application to Navier-Stokes equation
4. Noether's theorem

Notations

Let $d \in \mathbb{N}$, I be an open set in $\mathbb{R}$, and $a, b \in \mathbb{R}$, $a < b$, such that $[a, b] \subset I$.
Let $\mathcal{F}(I, \mathbb{R}^d)$ the set of functions defined in I taking value in $\mathbb{R}^d$.
Let $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) be the set of continuous functions $x : I \to \mathbb{R}^d$ (respectively $x : I \to \mathbb{C}^d$).
Let $n \in \mathbb{N}$, and $\mathcal{C}^n(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^n(I, \mathbb{C}^d)$) be the set of functions in $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) which are differentiable up to order n.

Hölderian functions Let $w \in \mathcal{C}^0(I, \mathbb{R}^d)$. Let $t \in I$.

1. w is Hölder of Hölder exponent α, $0 < \alpha < 1$, at point t if

$$\exists c > 0,\ \exists \eta > 0\, s.t.\, \forall t' \in I \mid t - t' \mid \leq \eta \Rightarrow \|w(t) - w(t')\| \leq c \mid t - t' \mid^\alpha,$$

   where $\|\cdot\|$ is a norm on $\mathbb{R}^d$.

2. w is α-Hölder and inverse Hölder with $0 < \alpha < 1$, at point t if

$$\exists c, C \in \mathbb{R}^{+*},\ c < C,\ \exists \eta > 0\, s.t.\, \forall t' \in I \mid t - t' \mid \leq \eta$$
$$c \mid t - t' \mid^\alpha \leq \|w(t) - w(t')\| \leq C \mid t - t' \mid^\alpha$$

Example

$H^\alpha(I, \mathbb{R}^d) := \{x \in \mathcal{C}^0(I, \mathbb{R}^d), x$ is $\alpha -$ Hölder and inverse Hölder$\}$.

Let $\mathcal{C}^{k \oplus \alpha}(I, \mathbb{C}^d) \subset \mathcal{C}^0(I, \mathbb{C}^d)$ defined by: Example

$$\mathcal{C}^{k \oplus \alpha}(I, \mathbb{C}^d) := \{x \in : \mathcal{C}^0(I, \mathbb{C}^d),\ x(t) := u(t) + w(t),$$
$$u \in \mathcal{C}^k(I, \mathbb{C}^d),\ w \in H^\alpha(I, \mathbb{C}^d)\}.$$

Calculus of variations (1)

We consider admissible Lagrangian functions L

$$\begin{array}{rcl} \mathrm{L} : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d & \rightarrow & \mathbb{C} \\ (t, x, v) & \mapsto & L(t, x, v) \end{array}$$

such that $\mathrm{L}(t, x, v)$ is holomorphic with respect to v, differentiable with respect to x.

Example $L(t, x, v) = \frac{1}{2} v^2 - U(x)$, with U is a function of x.

A Lagrangian function defines a *functional* on $\mathcal{C}^1(I, \mathbb{R})$, denoted by

$$\begin{array}{rcl} \mathcal{L} : \mathcal{C}^1(I, \mathbb{R}^d) & \rightarrow & \mathbb{R} \\ x & \longmapsto & \mathcal{L}(x) := \displaystyle\int_a^b \mathrm{L}\big(s, x(s), \frac{dx}{dt}(s)\big)\, ds\,. \end{array}$$

Let V be the space of variations defined by:

$$V := \{h \in \mathcal{C}^1(I, \mathbb{R}^d),\ h(a) = h(b) = 0\}.$$

Calculus of variations (2)

A functional $\mathcal{L}$ is differentiable at point $\gamma \in \mathcal{C}^2(I, \mathbb{R}^d)$ if and only if

$$\mathcal{L}(\gamma + \theta h) - \mathcal{L}(\gamma) = \theta D\mathcal{L}(\gamma)(h) + o(\theta),$$

for $\theta > 0$ sufficiently small and any $h \in V$. $D\mathcal{L}(\gamma)(h)$ is the Gâteaux derivative of $\mathcal{L}$ at point γ in the direction h.
An extremal for the functional $\mathcal{L}$ is a function $\gamma \in \mathcal{C}^2(I, \mathbb{R}^d)$ such that $D\mathcal{L}(\gamma)(h) = 0$ for any $h \in V$.

Theorem

The extremals of $\mathcal{L}$ coincide with the solutions of the Euler-Lagrange equation denoted by (EL) and defined by

$$\frac{d}{dt}\left[\frac{\partial \mathrm{L}}{\partial v}\left(t, \gamma(t), \frac{d\gamma}{dt}(t)\right)\right] = \frac{\partial \mathrm{L}}{\partial x}\left(t, \gamma(t), \frac{d\gamma}{dt}(t)\right). \qquad (EL)$$

Quantum derivative (1)

Idea: Extension of the classical notion of derivative. Let $x \in \mathcal{C}^0(I, \mathbb{R}^d)$. For all $\epsilon > 0$, we call ϵ-left and right quantum derivatives the quantities

$$d_\epsilon^+ x(t) := \frac{x(t+\epsilon) - x(t)}{\epsilon}, \quad d_\epsilon^- x(t) := \frac{x(t) - x(t-\epsilon)}{\epsilon},$$

Definition

For any $\epsilon > 0$, the ϵ-scale derivative of x at point t is the quantity defined for $\mu \in \{1, -1, 0, i, -i\}$ by

$$\begin{array}{rcl} \dfrac{\Box_\epsilon}{\Box t} : \mathcal{C}^0(I, \mathbb{R}^d) & \rightarrow & \mathcal{C}^0(I, \mathbb{C}^d) \\ x & \mapsto & \dfrac{\Box_\epsilon x}{\Box t} \end{array}$$

where $\dfrac{\Box_\epsilon x}{\Box t}(t) := \dfrac{1}{2}\Big[\big(d_\epsilon^+ x(t) + d_\epsilon^- x(t)\big) + i\mu\big(d_\epsilon^+ x(t) - d_\epsilon^- x(t)\big)\Big] \; \forall t \in I.$

Remarks

- If $x \in \mathcal{C}^1(I, \mathbb{R}^d)$, then $\lim_{\epsilon \to 0} \frac{\Box_\epsilon x}{\Box t} = \frac{dx}{dt}$ the classical derivative of x.
- For $\mu = i$, $\frac{\Box_\epsilon}{\Box t} = d_\epsilon^-$
- For $\mu = -i$, $\frac{\Box_\epsilon}{\Box t} = d_\epsilon^+$

$\rightarrow$ Allows to recover the backward and forward derivatives.

- Extension for $x \in \mathcal{C}^0(I, \mathbb{C}^d)$ by

$$\frac{\Box_\epsilon x}{\Box t}(t) := \frac{\Box_\epsilon \mathrm{Re}(x)}{\Box t} + i \frac{\Box_\epsilon \mathrm{Im}(x)}{\Box t}, \tag{1}$$

where $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ are the real and imaginary part of x.
$\rightarrow$ composition of $\frac{\Box_\epsilon}{\Box t}$

Quantum derivative 2

Idea: Build an analogous of the derivative for “non-differentiable” functions.

Construction

We consider $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$ the space of continuous functions

$$\begin{aligned} f : I\times]0,1] &\rightarrow \mathbb{R}^d \\ (t,\epsilon) &\mapsto f(t,\epsilon) \end{aligned}$$

Let $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ be a subspace of $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$:

$$\begin{aligned} \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d) := & \{f \in \mathcal{C}^0(I\times]0,1],\mathbb{R}^d), \\ & \lim_{\epsilon\rightarrow 0} f(t,\epsilon) \text{ exists for any } t \in I\}. \end{aligned}$$

Let E be a complementary space of $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ in $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$.

Let π be the projection onto $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ defined by

$$\begin{aligned} \pi : \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d) \oplus E &\rightarrow \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d) \\ f_{conv} + f_E &\mapsto f_{conv} \,. \end{aligned}$$

Quantum derivative 3

We can then define the operator $\langle\,.\,\rangle$ by

$$\begin{aligned}\langle\,.\,\rangle : \mathcal{C}^0(I\times]0,1],\mathbb{R}^d) &\rightarrow \mathcal{F}(I,\mathbb{R}^d)\\ f &\mapsto \langle\pi(f)\rangle : t\mapsto \lim_{\epsilon\rightarrow 0}\pi(f)(t,\epsilon)\,.\end{aligned}$$

Definition

Let us introduce the new operator $\frac{\Box}{\Box t}$ (without ϵ) on the space $\mathcal{C}^0(I,\mathbb{R}^d)$ by:

$$\frac{\Box x}{\Box t} := \langle\pi(\frac{\Box_\epsilon x}{\Box t})\rangle \qquad (2)$$

- For $x\in\mathcal{C}^1(I,\mathbb{R}^d)$, then $\frac{\Box x(t)}{\Box t} = \frac{dx}{dt}(t)$.
- For $w\in H^\alpha(I,\mathbb{R}^d)$, then $\frac{\Box w(t)}{\Box t} = 0$. (Since $c.\epsilon^{\alpha-1}\leq\|\Box_\epsilon w(t)\|$)
- For $x\in\mathcal{C}^{1\oplus\alpha}(I,\mathbb{C}^d)$, $0<\alpha<1$, with $x := u+w$, then $\frac{\Box x(t)}{\Box t} = u'(t)$.

Quantum derivative 4

Properties

- Non-differentiable Leibniz rule
  Let f be α-Hölder and g be β- Hölder, with $\alpha+\beta>1$,

$$\frac{\Box}{\Box t}(f\cdot g)=\frac{\Box f}{\Box t}\cdot g+f\cdot\frac{\Box g}{\Box t}\,.$$

- Composition
  Let f be a $\mathcal{C}^2(\mathbb{R}^d\times I,\mathbb{R})$ function. Let $\frac{1}{2}\leq\alpha<1$. Let $x=(x_1,\ldots,x_d)\in C^{1\oplus\alpha}(I,\mathbb{R}^d)$ written as $x:=u+w$ where $u=(u_1,\ldots,u_d)\in\mathcal{C}^1(I,\mathbb{R}^d)$ and $w=(w_1,\ldots,w_d)\in H^\alpha(I,\mathbb{R}^d)$, then the following formula holds

$$\begin{aligned}\frac{\Box f(x(t),t)}{\Box t} &= \nabla_x f(x(t),t)\cdot\nabla u(t)+\frac{\partial f}{\partial t}(x(t),t)\\ &\quad+\sum_{k=1}^{d}\sum_{l=1}^{d}\frac{1}{2}\frac{\partial^2 f}{\partial x_k\partial x_l}(x(t),t)\,a_{k,l}(w(t))\,,\end{aligned}$$

$$a_{k,l}(x(t)):=\langle\pi\Big(\tfrac{\epsilon}{2}\big((d_\epsilon^+x_k(t))(d_\epsilon^+x_l(t))(1+i\mu)-(d_\epsilon^-x_k(t))(d_\epsilon^-x_l(t))(1-i\mu)\big)\Big)\rangle.$$

Non-differentiable embedding of operators

1. We denote by O the differential operator acting on $\mathcal{C}^n(I, \mathbb{C}^d)$ defined by

$$\mathrm{O} = \sum_{i=0}^{n} \mathrm{F}_i \cdot \left(\frac{d^i}{dt^i} \circ \mathrm{G}_i\right), \tag{3}$$

   where $\cdot$ is the standard product of operators and $\circ$ the usual composition, *i.e.* $(A \circ B)(x) = A(B(x))$, with the convention that $\left(\frac{d}{dt}\right)^0 = \mathrm{Id}$, where Id denotes the identity mapping on $\mathbb{C}$.

2. The non-differentiable embedding of O written as (3), denoted by $\mathrm{Emb}_{\square}(\mathrm{O})$ is the operator

$$\mathrm{Emb}_{\square}(\mathrm{O}) = \sum_{i=0}^{n} F_i \cdot \left(\frac{\square^i}{\square t^i} \circ G_i\right). \tag{4}$$

Remark: In the rest of the talk we will consider curves $x \in \mathcal{C}^0(I, \mathbb{R}^d)$, such that $\frac{\square x}{\square t} \in \mathcal{C}^0(I, \mathbb{R}^d)$ or smooth enough.

Non-differentiable embedding of ODE

1. Let the ordinary differential equation associated to O be defined by

$$\mathrm{O}\Big(x, \frac{dx}{dt}, \ldots, \frac{d^k x}{dt^k}\Big) = 0, \text{ for any } x \in \mathcal{C}^{k+n}(I, \mathbb{C}). \tag{5}$$

2. The non-differentiable embedding of equation (5) is defined by

$$\mathrm{Emb}_{\Box}\,(\mathrm{O})\,(x, \frac{\Box x}{\Box t}, \ldots, \frac{\Box^k x}{\Box t^k}) = 0, \;\; x, \left(\frac{\Box^i x}{\Box t^i}\right)_{1 \le i \le k} \in \mathcal{C}^0(I, \mathbb{C}^d). \tag{6}$$

The non-differentiable embedded Euler-Lagrange equation

1. The Euler-Lagrange equation (EL) is:

$$\frac{d}{dt}\left[\frac{\partial \mathrm{L}}{\partial v}\left(t,\gamma(t),\frac{d\gamma}{dt}(t)\right)\right]=\frac{\partial \mathrm{L}}{\partial x}\left(t,\gamma(t),\frac{d\gamma}{dt}(t)\right). \qquad (EL)$$

2. Let $\mathrm{O}_{(EL)}$ be the associated non-differentiable embedded operator

$$\mathrm{O}_{(EL)}:=\frac{d}{dt}\circ\frac{\partial \mathrm{L}}{\partial v}-\frac{\partial \mathrm{L}}{\partial x}$$

   The Euler-Lagrange equation (EL) is

$$\mathrm{O}_{(EL)}((t,\gamma(t),\frac{d\gamma}{dt}(t))=0\,.$$

3. The non-differentiable Euler-Lagrange associated to (EL) is then:

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}(t,\gamma(t),\frac{\Box}{\Box t}\gamma(t))\right)-\frac{\partial L}{\partial x}(t,\gamma(t),\frac{\Box}{\Box t}\gamma(t))=0. \;\; \mathrm{Emb}_{\Box}(EL)$$

Embedding of the Lagrangian functional

The Lagrangian functional associated to L is:

$$\mathcal{L} : \mathcal{C}^1(I, \mathbb{R}^d) \to \mathbb{R}, \quad x \in \mathcal{C}^1(I, \mathbb{R}^d) \longmapsto \int_a^b \mathrm{L}\big(s, x(s), \frac{dx}{dt}(s)\big)\, ds\,.$$

The natural embedding of the Lagrangian functional $\mathcal{L}$ is given by

$$\mathcal{L}_\Box : \mathcal{C}^0(I, \mathbb{R}^d) \to \mathbb{R}, \quad x \in \mathcal{C}^0(I, \mathbb{R}^d) \longmapsto \int_a^b \mathrm{L}\big(s, x(s), \frac{\Box x(s)}{\Box t}\big)\, ds\,,$$

always with $\frac{\Box x}{\Box t} \in \mathcal{C}^0(I, \mathbb{R}^d)$.

Non-differentiable calculus of variations

Let α, β be real numbers $0 < \alpha, \beta < 1$, s.t. $\alpha + \beta > 1$.
Let $V := \{h \in \mathcal{C}^0(I, \mathbb{R}^d), \beta$-Hölder , $h(a) = h(b) = 0\}$, be the space of non-differentiable variations.

Definition

Let $\Phi : \mathcal{C}^0(I, \mathbb{R}^d) \rightarrow \mathbb{C}$ be a functional. The functional Φ is called V-differentiable on a curve $\gamma \in \mathcal{C}^0(I, \mathbb{R}^d)$, α-Hölder if and only if its Gâteaux differential

$$\lim_{\epsilon \longrightarrow 0} \frac{\Phi(\gamma + \epsilon h) - \Phi(\gamma)}{\epsilon}$$

exists in any direction $h \in V$. And then $D\Phi$ is called its differential and is given by

$$D\Phi(\gamma)(h) = \lim_{\epsilon \longrightarrow 0} \frac{\Phi(\gamma + \epsilon h) - \Phi(\gamma)}{\epsilon}.$$

V-extremal curves

A V-extremal curve of the functional Φ on the space V of curves is a curve γ α-H ölder satisfying

$$D\Phi(\gamma)(h) = 0, \text{for any } h \in V.$$

Theorem

The differential of $\mathcal{L}_\square$ on $\gamma \in \mathcal{C}^0(I, \mathbb{R}^d)$, α-Hölder and $\frac{\square\gamma}{\square t}$ α-Hölder is given for any $h \in V$ by

$$D\mathcal{L}_\square(\gamma)(h) = \int_a^b \left(\frac{\partial L}{\partial x}\Big(t, \gamma(t), \frac{\square\gamma(t)}{\square t}\Big) \cdot h(t) + \frac{\partial L}{\partial v}\big(t, \gamma(t), \frac{\square\gamma(t)}{\square t}\big) \cdot \frac{\square h(t)}{\square t} \right) dt.$$

Theorem (Non-differentiable least-action principle)

Let $0 < \alpha < 1$, $\alpha + \beta > 1$ and $\beta \leq \alpha$. Let L be an admissible Lagrangian function of class $\mathcal{C}^2$. We assume that $\gamma \in \mathcal{C}^0(I, \mathbb{R}^d)$ α-Hölder and $\frac{\Box\gamma}{\Box t}$ α-Hölder. The curve γ is an extremal curve of the functional $\mathcal{L}_\Box$ on the space of variations V, if and only if it satisfies the following generalized Euler-Lagrange equation

$$\frac{\partial L}{\partial x}\left(t, \gamma(t), \frac{\Box\gamma(t)}{\Box t}\right) - \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\left(t, \gamma(t), \frac{\Box\gamma(t)}{\Box t}\right)\right) = 0 \,. \qquad (NDEL)$$

Coherence

- Embedding of the Euler-Lagrange equation denoted by Emb(EL).
- Embedding of the Lagrangian functional $\mathcal{L}_{\square}$.
- Non-differentiable calculus of variation $\longrightarrow$ leads to a non-differentiable Euler-Lagrange equation N.D EL.

Conclusion N.D EL = Emb(EL). We preserve the Lagrangian structure passing to the non-differentiable embedding.

Application to the Navier-Stokes equation

Extension of the definition of characteristics

The classical method of characteristics for a PDE is to look for $t \to x(t)$ satisfying the following ordinary differential equation

$$\frac{d}{dt}\left(u(x(t),t)\right) = F(x(t),t),$$

where F is the non homogeneous part of the PDE.
Using the operator $\frac{\Box}{\Box t}$ one can generalize this method. We say that a curve $t \to x(t)$ is a non-differentiable characteristic for a given PDE if the solution $u(x(t),t)$ satisfies

$$\frac{\Box}{\Box t}\left(u(x(t),t) = F(x(t),t),\right.$$

and x and t satisfy an ordinary differential equation in $\frac{\Box}{\Box t}$.

Let us consider the Navier-Stokes equation

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{d} u_k \frac{\partial u}{\partial x_k} = \nu \Delta_x u - \nabla_x p,$$

where the unknown are the velocity $u(t,x) \in \mathbb{R}^d$, $u = (u_1, \dots, u_d)$, and the pressure $p(t,x) \in \mathbb{R}$. The constant $\nu \in \mathbb{R}^+$ is the viscosity.

Theorem

The non-differentiable characteristics $x \in \mathcal{C}^{1\oplus\alpha}_{\text{nav}}$, $\frac{1}{2} \leq \alpha < 1$ of the Navier-Stokes equations correspond to $\mathcal{C}^{1\oplus\alpha}_{\text{nav}}$ extremals of the Lagrangian

$$L(t,x,v) = \frac{1}{2} v^2 - p(x,t),$$

$$\mathcal{C}^{1\oplus\alpha}_{\text{nav}} := \{x = (x_1, \dots, x_d) \in \mathcal{C}^{1\oplus\alpha}(I, \mathbb{R}^d),\ x_i(t) = \int_0^t u_i(x(s), s)\, ds + W_i(t),$$

$$W_i \in H^\alpha, \frac{1}{2} \leq \alpha < 1,\ i = 1, \dots, d\},$$

where u is a solution of the Navier-Stokes equation and W satisfies

$$a_{l,l}(W(t)) = -2\nu \quad \textit{and} \quad a_{k,l}(W(t)) = 0 \text{ if } k \neq l.$$

Idea of the proof

For $x \in \mathcal{C}^{1\oplus\alpha}_{\text{nav}}(I, \mathbb{R}^d)$ we have $\frac{\Box x}{\Box t}(t) = u(x(t), t)$, and for any $i = 1, \ldots, d$

$$\frac{\Box u_i(x(t), t)}{\Box t} = \nabla_x u_i(x(t), t) \cdot u_i(x(t), t) + \frac{\partial u_i}{\partial t}(x(t), t) + \sum_{k=1}^{d} \sum_{l=1}^{d} \frac{1}{2} \frac{\partial^2 u_i}{\partial x_k \partial x_l}(x(t), t) a_{k,l}(w(t)).$$

The non-differentiable caracteristics are curve $t \to x(t)$ such that

$$\frac{\Box}{\Box t}\left(u(x(t), t)\right) = -\nabla_x p.$$

this equation can be rewritten as

$$\frac{\Box}{\Box t}\left(\frac{\Box x}{\Box t}\right) = -\nabla_x p.$$

which is the non-differentiable Euler-Lagrange equation associated to L.

Noether's theorem (1)

- We call $\{\phi_s\}_{s\in\mathbb{R}}$ a one parameter group of diffeomorphisms $\phi_s : \mathbb{R}^d \to \mathbb{R}^d$, of class $\mathcal{C}^1$ satisfying

$$\phi_0 = \mathrm{Id}, \quad \phi_s \circ \phi_u = \phi_{s+u}, \quad \phi_s \text{ is of class } \mathcal{C}^1 \text{ with respect to } s.$$

- Invariance
  Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be invariant under the action of Φ if

$$L\Big(t, x(t), \frac{dx}{dt}(t)\Big) = L\Big(t, \phi_s(x(t)), \frac{d}{dt}\big(\phi_s(x(t))\big)\Big), \ \ \forall s \in \mathbb{R}, \forall t \in \mathbb{R},$$

  for any solution x of the Euler-Lagrange equation.

Noether's theorem (2)

- First Integral
  A first integral for the Euler-Lagrange equation is a function $J : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that for any solution x of the Euler-Lagrange equation,

$$\frac{d}{dt}\big(J(t, x(t), \dot{x}(t))\big) = 0 \quad \text{for any } t \in \mathbb{R}.$$

Noether's theorem Let L be an admissible Lagrangian of class $\mathcal{C}^2$ invariant under $\Phi = \{\phi_s\}_{s \in \mathbb{R}}$, a one parameter group of diffeomorphisms. Then, the function

$$J : (t, x, v) \mapsto \frac{\partial L}{\partial v}(t, x, v) \cdot \frac{d\phi_s(x)}{ds} \;|_{s=0}$$

is a first integral of the Euler-Lagrange equation (EL).

Passage to the non-differentiable case?

invariance of Lagrangian

Noether's thm. $\Big\downarrow$

First integral

Passage to the non-differentiable case?

$$\begin{array}{ccc} \text{invariance of Lagrangian} & \xrightarrow{\text{N.D. Emb}} & \text{invariance of N.D Lagrangian} \\ \text{Noether's thm.}\downarrow & & \\ \text{First integral} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. First integral} \end{array}$$

Passage to the non-differentiable case?

$$\begin{array}{ccc}
\text{invariance of Lagrangian} & \xrightarrow{\text{N.D. Emb}} & \text{invariance of N.D Lagrangian} \\
\text{Noether's thm.} \Big\downarrow & & \Big\downarrow \text{N.D. Noether's thm.} \\
\text{First integral} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. First integral}
\end{array}$$

Non-differentiable Noether's theorem (1)

- $\Box$-invariance Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be $\Box$-invariant under the action of Φ if

$$L(t, x(t), \frac{\Box x}{\Box t}(t)) = L(t, \phi_s(x(t)), \frac{\Box}{\Box t}(\phi_s(x(t)))), \quad \forall s \in \mathbb{R}, \quad \forall t \in I.$$

for any solution $x \in \mathcal{C}^1_\Box$ of the non-differentiable Euler-Lagrange equation (NDEL).

- Persistence of invariance?
- Sufficient condition:

Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms, such that $\phi_s : \mathbb{C}^d \to \mathbb{C}^d$ satisfies the $\Box$-commutation property:

$$\frac{\Box}{\Box t}(\phi_s(x)) = \phi_s\left(\frac{\Box x}{\Box t}\right), \quad \forall s \in \mathbb{R}. \tag{7}$$

If L is strongly invariant i.e:

$$L(t, x, v) = L(t, \phi_s(x), \phi_s(v)), \ \ \forall s \in \mathbb{R}, \ \forall t \in I, \ \forall x \in \mathbb{R}^d, \ \forall v \in \mathbb{R}^d.$$

Then, L is $\Box$-invariant under the action of $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$.

Non-differentiable Noether's theorem (2)

- Let ϕ be a linear map, then ϕ satisfies the property of $\Box$-commutation.
- A generalized first integral associated to the non-differentiable Euler-Lagrange equation is a function $J : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d \to \mathbb{C}$ such that for any solution x of (NDEL), we have

$$\frac{\Box}{\Box t}\Big(J\big(t, x(t), \frac{\Box x(t)}{\Box t}\big)\Big) = 0 \qquad \forall t \in \mathbb{R}.$$

Non-differentiable Noether's theorem Let L be a Lagrangian of class $\mathcal{C}^2$ $\Box$-invariant under $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$, a one parameter group of diffeomorphisms, such that $\phi_s : \mathbb{C}^d \to \mathbb{C}^d$, for any $s \in \mathbb{R}$. Then, the function

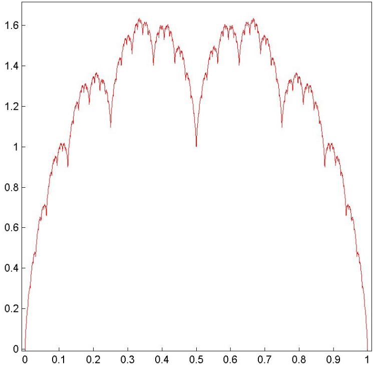
$$J : (t, x, v) \mapsto \frac{\partial L}{\partial v}(t, x, v) \cdot \frac{d\phi_s(x)}{ds} \; |_{s=0} \tag{8}$$

is a generalized first integral of the non-differentiable Euler-Lagrange equation (NDEL) on $H^\alpha(I, \mathbb{R}^d)$ with $\frac{1}{2} < \alpha < 1$.

Conclusion

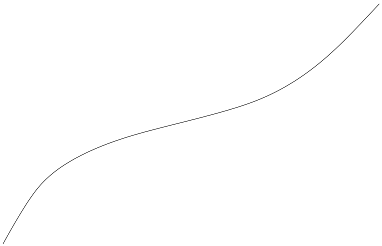
- The non-differential embedding preserves the Lagrangian structure
- Solutions of the Navier-Stokes seen as extremals of a non-differentiable Lagrangian
- Coherence for the Hamiltonian systems
- Persistence of the invariance of the Lagrangian under special conditions.
- Non-differentiable Noether theorem

Example of inverse-Hölder function: Tagaki-Knopp



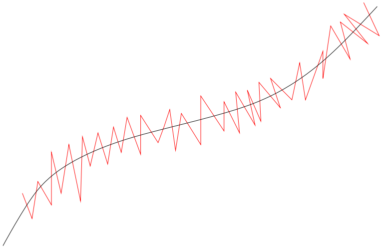
Retour

Example of smooth curve



Retour

Example of non-smooth curve



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Outline– Diagram

Lagrangian functional $\mathcal{L}$

least-action principle $\Big\downarrow$

Newton's equation

▸ Retour

Outline– Diagram

Lagrangian functional $\mathcal{L}$

least-action principle $\downarrow$

Newton's equation $\xrightarrow{\text{N.D. Emb}}$ N.D. Newton's equation

▸ Retour

Outline– Diagram

$$\begin{array}{ccc} \text{Lagrangian functional } \mathcal{L} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Lagrangian functional } \mathcal{L}_{\Box} \\ {\scriptstyle\text{least-action principle}}\downarrow & & \\ \text{Newton's equation} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Newton's equation} \end{array}$$

Retour

Outline– Diagram

$$\begin{array}{ccc}
\text{Lagrangian functional } \mathcal{L} & \xrightarrow{\text{N.D. Emb}} & \text{N.D Lagrangian functional } \mathcal{L}_{\Box} \\
\text{least-action principle} \Big\downarrow & & \Big\downarrow \text{N.D. least-action principle} \\
\text{Newton's equation} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Newton's equation}
\end{array}$$

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