Non parametric estimation of Archimedean copulas and tail dependence

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Part I. Transformations of Archimedean copulas

Copulas

Let F be a $d-$dimensional cdf with marginals F_i, $i = 1, \ldots, d$:
By Sklar's theorem, there exists a copula function $C : [0,1]^d \to [0,1]$ that links the distribution F with its margins:

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) .$$

- C is a $d-$dimensional cdf on $[0,1]^d$ with uniform marginals.
- C is unique if marginals F_i are continuous.

Archimedean copulas

Archimedean copulas:

$$C(u_1, \ldots, u_d) = \phi\left(\phi^{-1}(u_1) + \ldots + \phi^{-1}(u_d)\right)$$

$\phi : \mathbb{R}^+ \to [0,1]$ is the generator of the Archimedean copula,
$\phi^{-1}(x) = \inf\{s \in \mathbb{R}^+ : \phi(s) \leq x\}$ its (generalized) inverse function.

Generator: ϕ is continuous, decreasing, $d-$monotone (cf. McNeil and Nešlehová, 2009), $\phi(0) = 1$, $\lim_{x \to +\infty} \phi(x) = 0$.

One limitation: we consider only here strict generators, i.e. $\forall x \in \mathbb{R}^+$, $\phi(x) > 0$. $\Rightarrow$ ϕ is strictly decreasing and ϕ^{-1} is the regular inverse of ϕ.

Proposition (Equivalent generators, cf. Nelsen)

Generator $\phi_a(x) = \phi(ax)$ and $\phi(x)$ lead to the same copula, $a \in \mathbb{R} \setminus \{0\}$

implies that one can ask ϕ to be such that $\phi(t_0) = \varphi_0$ for an arbitrary point $(t_0, \varphi_0) \in \mathbb{R} \times (0,1)$.

Transformed Archimedean copulas

Proposition (Transformed copula)

Copula $\widetilde{C}$ depends on a continuous increasing function $T : [0,1] \to [0,1]$,

$$\widetilde{C}(u_1, \ldots, u_d) = T(C_0(T^{-1}(u_1), \ldots, T^{-1}(u_d))).$$

and if the initial C_0 is a given Archimedean copula with generator ϕ_0, then $\widetilde{C}$ is an Archimedean copula with generator

$$\widetilde{\phi}(x) = T \circ \phi_0(x)$$

Admissibility conditions for $\widetilde{\phi}$ (d-monotonicity, cf. McNeil and Nešlehová, 2009)
$\Rightarrow$ **more admissibility conditions for** T (e.g. using Faa Di Bruno formula).

Literature on these transformations: Durrleman and al. (2000), Charpentier (2008), Valdez and Xiao (2011)

Motivations

Possibility to transform both copulas and margins:

$$\widetilde{F}(x_1, \ldots, x_d) = \widetilde{C}(\widetilde{F}_1(x_1), \ldots, \widetilde{F}_d(x_d)),$$

where $\widetilde{F}_i = T \circ T_i^{-1} \circ F_i$ and $T_i : [0,1] \to [0,1]$ are continuous and increasing.

Why using transformations (instead of parametric multivariate distributions)?

1. **Flexible: allows distortions composition**
   - huge variety of reachable distributions (multimodal, etc.)
   - possibility to improve a fit gradually
2. **Invertible: analytical expressions**
   - for the expression of the distribution function;
   - but also for the expression of level curves;
3. **Estimation facilities**

Part II. Estimation of the distorted Archimedean copula

→ **Self-nested diagonals**

- Non-parametric estimation
- Parametric estimation

Diagonal of the copula

Idea: building a non-parametric estimator of the generator ϕ based on the diagonal section of the copula.

Definition (Diagonal section of the copula)

Consider a copula C satisfying *regular conditions*. For all $u \in [0,1]$,

$$\delta_1(u) = C(u, \ldots, u)\,,$$

and δ_{-1} is the inverse function of δ_1 so that $\delta_1 \circ \delta_{-1} = \mathrm{I_d}$.

Remark:

- The copula C is not *always* uniquely determined by its diagonal (cf. Frank's condition, Erdely and al. 2013)
- Estimation based *only* on this diagonal may fail to capture tail dependence when $\phi'(0) = -\infty$.

Definitions

Definition (Discrete self-nested diagonals)

Consider a copula C satisfying *regular conditions*. The *discrete self-nested diagonal* of C at order k is the function δ_k such that for all $u \in [0,1]$, $k \in \mathbb{N}$

$$\begin{cases} \delta_k(u) & = & \delta_1 \circ \ldots \circ \delta_1(u), \quad (k \text{ times}), \\ \delta_{-k}(u) & = & \delta_{-1} \circ \ldots \circ \delta_{-1}(u), \quad (k \text{ times}), \\ \delta_0(u) & = & u, \end{cases}$$

where $\delta_1(u) = C(u, \ldots, u)$ and δ_{-1} is the inverse function of δ_1, so that $\delta_1 \circ \delta_{-1}$ is the identity function.

Definition (Self-nested diagonals)

Functions of a family $\{\delta_r\}_{r\in\mathbb{R}}$ are called (extended) self-nested diagonals of a copula C, if $\delta_k(u)$ is the discrete self-nested diagonal of C at order k, for all $k \in \mathbb{Z}$, and if furthermore

$$\delta_{r_1+r_2}(u) \quad = \quad \delta_{r_1} \circ \delta_{r_2}(u), \quad \forall\, r_1, r_2 \in \mathbb{R}, \forall\, u \in [0,1].$$

Self-nested diagonals

Proposition (Self-nested diagonals of an Archimedean copula)

If C is an Archimedean copula associated with a generator ϕ, then the self-nested diagonal of C at order r is

$$\delta_r(x) \quad = \quad \phi(d^r \cdot \phi^{-1}(x)), \qquad r \in \mathbb{R}.$$

Proposition (Interpolation of self-nested diagonals)

Let C be an Archimedean copula with generator ϕ. For any real $r \in [k, k+1]$, $k \in \mathbb{Z}$, the self-nested diagonals of C satisfies:

$$\delta_r(x) \quad = \quad \phi\left(\left(\phi^{-1} \circ \delta_k(x)\right)^{1-\alpha} \left(\phi^{-1} \circ \delta_{k+1}(x)\right)^{\alpha}\right), \qquad x \in [0,1],$$

with $\alpha = r - \lfloor r \rfloor$ and $k = \lfloor r \rfloor$, where $\lfloor r \rfloor$ denotes the integer part of r. Interpolation does not depend on the Gumbel parameter in the Gumbel case.

Expression of distortions using self-nested diagonals

Proposition (Distortion T using self-nested diagonals)

Consider Archimedean copulas C_0 and $\widetilde{C}$ satisfying regular conditions and the associated self-nested diagonals δ_r and $\widetilde{\delta}_r$, $r \in \mathbb{R}$. If T is defined by $T(0) = 0$, $T(1) = 1$ and for all $x \in (0,1)$,

$$T(x) = \widetilde{\delta}_{r(x)}(y_0),$$
$$\text{with } r(x) \text{ such that } \delta_{r(x)}(x_0) = x,$$

then the distorted copula using distortion T is equal to $\widetilde{C}$: for all $u_1, \ldots, u_d$,

$$\widetilde{C}(u_1, ..., u_d) = T \circ C_0(T^{-1}(u_1), \ldots, T^{-1}(u_d)),$$

where $(x_0, y_0) \in (0,1)^2$ can be arbitrarily chosen. In the case where C_0 is the independence copula,

$$r(x) = \frac{1}{\ln d} \ln\left(\frac{-\ln x}{-\ln x_0}\right),$$

Expression of generators using self-nested diagonals

Proposition (Generator $\widetilde{\phi}$ using self-nested diagonals)

Consider an Archimedean copula $\widetilde{C}$ satisfying regular conditions, and the associated self-nested copulas $\widetilde{\delta}_r$, for $r \in \mathbb{R}$. Assume that the copula $\widetilde{C}$ is reachable by distorting an Archimedean copula C_0, and denote by δ_r, $r \in \mathbb{R}$, the self-nested diagonals of C_0 and by ϕ_0 its generator. A generator $\widetilde{\phi}$ of $\widetilde{C}$ is defined for all $t \in \mathbb{R}^{*+}$ by

$$\widetilde{\phi}(t) = \widetilde{\delta}_{\rho(t)}(y_0)\,,$$
$$\text{with } \rho(t) \text{ such that } \delta_{\rho(t)}(x_0) = \phi_0(t)\,,$$

where $(x_0, y_0) \in (0,1)^2$ can be arbitrarily chosen. In the particular case where C_0 is the independent copula, then

$$\rho(t) = \frac{1}{\ln d} \ln\left(\frac{t}{-\ln x_0}\right)$$

Part II. Estimation of the distorted Archimedean copula

- Self-nested diagonals
- → **Non-parametric estimation**
- Parametric estimation

Non-parametric estimators of self-nested diagonals I

- First, build a (smooth) estimator $\widehat{\delta}_1$ of the diagonal of the target distorted copula $\widetilde{C}$ (e.g. empirical copula, Deheuvels (1979), Fermanian and al. (2004), Omelka and al. (2009)), and denote its inverse function $\widehat{\delta}_{-1}$.
- Estimators of *discrete self-nested diagonal* of $\widetilde{C}$ at order k are the function $\widehat{\delta}_k$ such that for all $u \in [0,1]$, $k \in \mathbb{N}$

$$\begin{cases} \widehat{\delta}_k(u) = \widehat{\delta}_1 \circ \ldots \circ \widehat{\delta}_1(u), & (k \text{ times}), \\ \widehat{\delta}_{-k}(u) = \widehat{\delta}_{-1} \circ \ldots \circ \widehat{\delta}_{-1}(u), & (k \text{ times}), \\ \widehat{\delta}_0(u) = u, & \end{cases}$$

Non-parametric estimators of self-nested diagonals II

- Define self-nested diagonals at non-integer orders with a *given* interpolation function z,

$$\widehat{\delta}_r(x) = z\left(\left(z^{-1} \circ \widehat{\delta}_k(x)\right)^{1-\alpha} \left(z^{-1} \circ \widehat{\delta}_{k+1}(x)\right)^{\alpha}\right), \quad x \in [0,1],$$

with $\alpha = r - \lfloor r \rfloor$ and $k = \lfloor r \rfloor$, where $\lfloor r \rfloor$ denotes the integer part of r.

Perfect interpolation functions z are functions such that for all $r_1, r_2 \in \mathbb{R}$,

$$\delta_{r_1} \circ \delta_{r_2} = \delta_{r_1+r_2}.$$

They do not depend on the Gumbel parameter in the Gumbel case (and in particular in Independent case).

Non-parametric estimators of $\widetilde{\phi}$

Definition (Non-parametric estimation of $\widetilde{\phi}$ - Case C_0 independent copula)

Consider an Archimedean copula $\widetilde{C}$ and associated self-nested diagonals $\widetilde{\delta}_r$, for $r \in \mathbb{R}$. Denote by $\widehat{\delta}_r$ the estimator of $\widetilde{\delta}_r$, for $r \in \mathbb{R}$. Assume that $\widetilde{\phi}(t_0) = \varphi_0$, for a given couple of values $(t_0, \varphi_0) \in \mathbb{R}^+ \setminus \{0\} \times (0,1)$. A non-parametric estimator $\widehat{\phi}$ of $\widetilde{\phi}$ is defined by $\widehat{\phi}(0) = 1$ and for all $t \in \mathbb{R}^+ \setminus \{0\}$,

$$\widehat{\phi}(t) = \widehat{\delta}_{\rho(t)}(\varphi_0)\,,$$

$$\text{with } \rho(t) = \frac{1}{\ln d} \ln\left(\frac{t}{t_0}\right)\,,$$

where $(t_0, \varphi_0) \in \mathbb{R}^+ \setminus \{0\} \times (0,1)$ can be arbitrarily chosen.

In particular, the estimator $\widehat{\phi}$ of $\widetilde{\phi}$ is passing through the points

$$\{(t_k, \varphi_k)\}_{k \in \mathbb{Z}} = \{(d^k t_0,\, \widehat{\delta}_k(\varphi_0))\}_{k \in \mathbb{Z}},$$

No interpolation function z is needed to get (t_k, φ_k), for $k \in \mathbb{Z}$.

Non-parametric estimators of T and $\tilde{\phi}$

Same kind of estimator for $\widehat{T}$ or $\widehat{\phi}$ in the general case:

$$\widehat{T}(x) = \widehat{\delta}_{r(x)}(y_0),$$
$$\text{with } r(x) \text{ such that } \delta_{r(x)}(x_0) = x,$$

where $(x_0, y_0) \in (0,1)^2$ can be arbitrarily chosen.

Proposition (Theoretical confidence bands for estimators - not detailed here)

Theoretical confidence bands on $\widehat{\delta}_1 \quad \Rightarrow \quad$ *theoretical confidence bands on* $\widehat{\phi}$.

one needs the distribution of the empirical process $\widehat{\delta}(u)$, $u \in [0,1]$.

Non-parametric $\widehat{\phi}(t)$

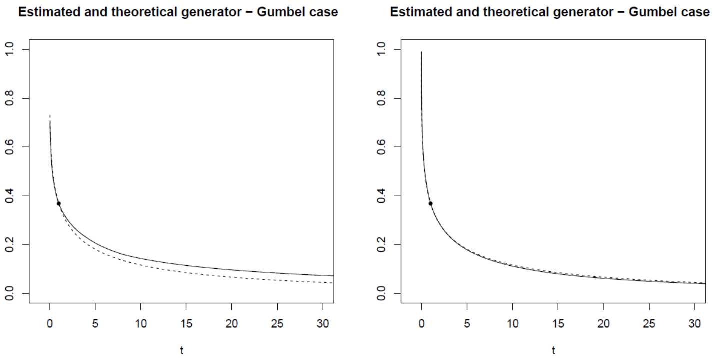
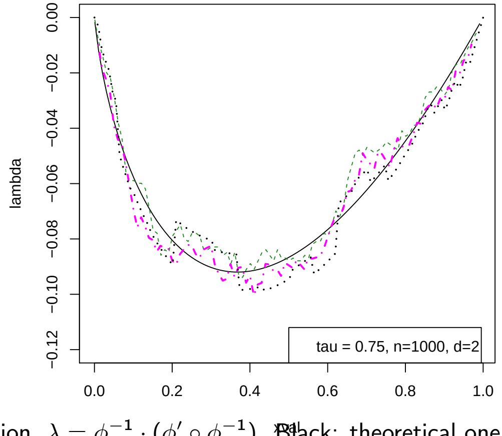


Figure : Estimated versus theoretical Gumbel-generator with parameter $\theta = 3$. Size of simulated samples $n = 150$ (left) and $n = 1500$ (right). Estimated $\widehat{\phi}(t) = \widehat{\delta}_{\rho(\mathbf{t})}(y_0)$ (full line). The theoretical standardized Gumbel-generator, i.e., $\bar{\phi}(t) = \exp(-t^{1/\theta})$, is drawn using a dashed line. We force the generators to pass through the point $(t_0, \varphi_0) = (1, e^{-1})$ (black point).

Comparison with other estimators



Estimation of λ function, $\lambda = \phi^{-1} \cdot (\phi' \circ \phi^{-1})$. Black: theoretical one. Dark green dashed line: estimator proposed by Genest and al. (2011), Pink and dotted lines: our estimator with two differentiation techniques.

Confidence bands for $\widehat{\delta_1}$ imply confidence bands for $\widehat{\phi}$

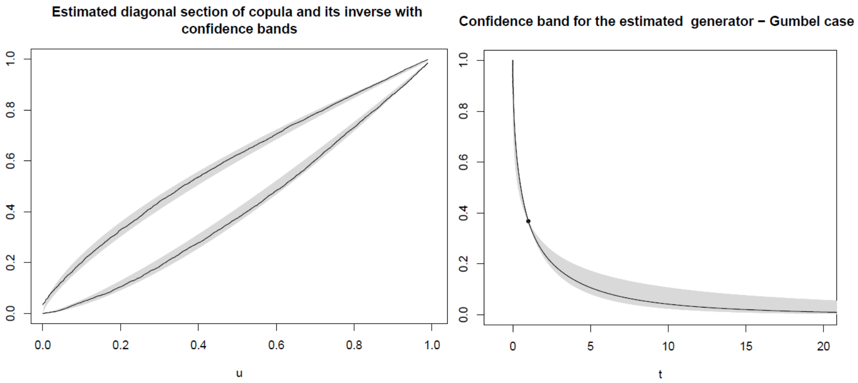


Figure : (Left) Confidence bands for $\widehat{\delta_1}$ and $\widehat{\delta_{-1}}$ for chosen parameters $\alpha^- = \beta^- = 1.05$, $\alpha^+ = \beta^+ = 0.9$. (Right) Resulting theoretical confidence band for $\widehat{\phi}$. Here $\widetilde{C}$ is a Gumbel copula of parameter $\theta = 2$, the size of generated sample is $n = 2000$.

Part II. Estimation of the distorted Archimedean copula

- Self-nested diagonals
- Non-parametric estimation

→ **Parametric estimation**

A class of parametric transformation

We take back from Bienvenüe and R. (2012):

Definition (Conversion and distortion functions)

Let f any bijective increasing function from $\mathbb{R}$ to $\mathbb{R}$. It is said to be a conversion function. The distortion $T_f : [0,1] \to [0,1]$ is defined as

$$T_f(u) = \begin{cases} 0 & \text{if } u = 0, \\ \mathrm{logit}^{-1}(f(\mathrm{logit}(u))) & \text{if } 0 < u < 1, \\ 1 & \text{if } u = 1. \end{cases}$$

Remark: Distortions function are chosen in a way to be easily invertible ($T_f \circ T_g = T_{f \circ g}$, $T_f^{-1} = T_{f^{-1}}$).

- We will use (composited) hyperbolic conversion functions:

$$f(x) = H_{m,h,\rho_1,\rho_2,\eta}(x) = m - h + (e^{\rho_1} + e^{\rho_2})\frac{x-m-h}{2} - (e^{\rho_1} - e^{\rho_2})\sqrt{\left(\frac{x-m-h}{2}\right)^2 + e^{\eta - \frac{\rho_1+\rho_2}{2}}}$$

with $m, h, \rho_1, \rho_2 \in \mathbb{R}$, and one smoothing parameter $\eta \in \mathbb{R}$.

- $H^{-1}_{m,h,\rho_1,\rho_2,\eta}(x) = H_{m,-h,-\rho_1,-\rho_2,\eta}(x)$.

Parametric distortions estimation

Framework:

- An initial given copula C_0 is distorted using a distortion T,
- Initial given margins are distorted using distortions $T_1, \ldots, T_d$.

Estimation:

- Get non-parametric estimators for T,
- Get non-parametric estimators for $T_1, \ldots, T_d$,
- Fit all distortions $T, T_1, \ldots, T_d$ by piecewise-linear functions (e.g. in the logit scale),
- Approach piecewise linear functions by composition of hyperbolas, cf. Bienvenüe and R. (2012).

Geyser data: distorted bivariate density

Non-parametric ⇒ Parametric estimation (without optimization).

Data : 272 eruptions of the **Old Faithful geyser** in Yellowstone National Park. Each observation consists of two measurements: the duration (in min) of the eruption (X), and the waiting time (in min) before the next eruption (Y).

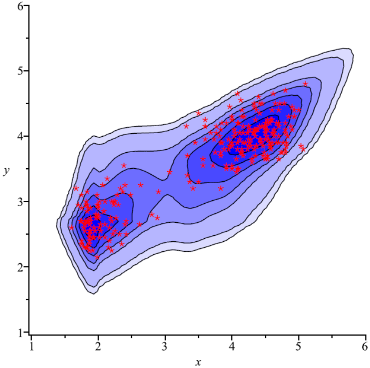


Figure : **Level curves of distorted density $\tilde{f}(x_1, x_2)$ and Old Faithful geyser data (red points.**

Part III. Tail dependence

→ **Definitions**

- Transformed Archimedean copulas
- Estimation given tail coefficients

multivariate TDC

Definition (Multivariate tail dependence coefficients: TDC)

Assume that the considered copula C is the distribution of some random vector $\mathbf{U} := (U_1, \ldots, U_d)$. Denote $I = \{1, \ldots, d\}$ and consider two non-empty subsets $I_h \subset I$ and $\bar{I}_h = I \setminus I_h$ of respective cardinal $h \geq 1$ and $d - h \geq 1$. A multivariate version of classical bivariate tail dependence coefficients is given by De Luca and Riviecio, 2012 (when limits exist):

$$\lambda_L^{I_h,\bar{I}_h} = \lim_{u \to 0^+} \lambda_L^{I_h,\bar{I}_h}(u) \quad \text{with} \quad \lambda_L^{I_h,\bar{I}_h}(u) = \mathbb{P}\left[U_i \leq u, i \in I_h \mid U_i \leq u, i \in \bar{I}_h\right],$$

$$\lambda_U^{I_h,\bar{I}_h} = \lim_{u \to 1^-} \lambda_U^{I_h,\bar{I}_h}(u) \quad \text{with} \quad \lambda_U^{I_h,\bar{I}_h}(u) = \mathbb{P}\left[U_i \geq u, i \in I_h \mid U_i \geq u, i \in \bar{I}_h\right].$$

If for all $I_h \subset I$, $\lambda_L^{I_h,\bar{I}_h} = 0$, (*resp.* $\lambda_U^{I_h,\bar{I}_h} = 0$) then we say $\mathbf{U}$ is lower tail independent (*resp.* upper tail independent).

With exchangeable r.v., depends on $d = \mathrm{card}(I)$ and $h = \mathrm{card}(I_h)$.
In particular case of Archimedean copulas, one have ($\psi = \phi^{-1}$):

Proposition (Multivariate Tail Dep. Coeffs. for Archimedean copulas)

For Archimedean copulas the multivariate lower and upper tail dependence coefficients are respectively (cf. De Luca and Riviecio, 2012):

$$\lambda_L^{(h,d-h)} = \lim_{u \to 0^+} \frac{\psi^{-1}(d\,\psi(u))}{\psi^{-1}((d-h)\,\psi(u))},$$

$$\lambda_U^{(h,d-h)} = \lim_{u \to 1^-} \frac{\sum_{i=0}^{d} (-1)^i\, C_d^i\, \psi^{-1}(i\,\psi(u))}{\sum_{i=0}^{d-h} (-1)^i\, C_{d-h}^i\, \psi^{-1}(i\,\psi(u))}.$$

Some references on tail dependence and Archimedean copulas: Juri and Wüthrich (2003), Charpentier and Segers (2009), Durante and al (2010).

TDC for some usual copulas

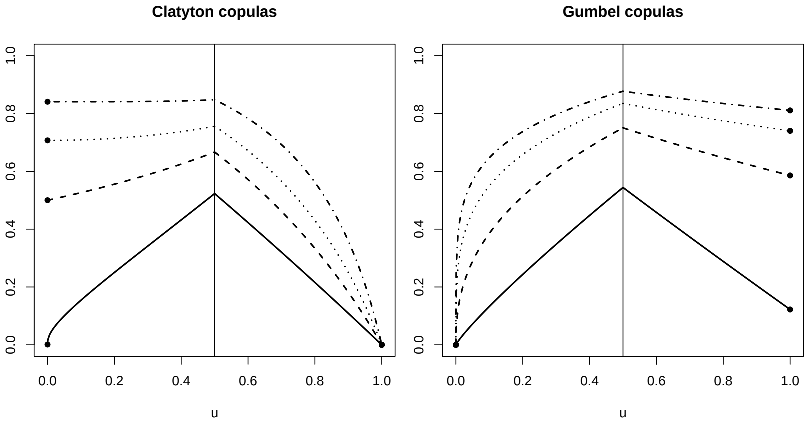


Figure : Shape of the *concentration* function $\lambda_{LU}(u) = \mathbf{1}_{\{u \le 1/2\}} \lambda_L(u) + \mathbf{1}_{\{u > 1/2\}} \lambda_U(u)$ for some Clayton copulas (left panel), Gumbel copulas (right panel).

Regular Variation

At infinity:

$$f \in \mathcal{RV}_\alpha(\infty) \;\Leftrightarrow\; \forall\, s > 0,\; \lim_{x \to +\infty} \frac{f(s\,x)}{f(x)} = s^\alpha.$$

At zero, using $M(x) = 1 - x$ and $I(x) = 1/x$,

$$f \in \mathcal{RV}_\alpha(0) \Leftrightarrow f \circ I \in \mathcal{RV}_{-\alpha}(\infty) \Leftrightarrow \forall\, s > 0,\; \lim_{x \to \mathbf{0}^+} \frac{f(s\,x)}{f(x)} = s^\alpha.$$

At one,

$$f \in \mathcal{RV}_\alpha(1) \Leftrightarrow f \circ M \circ I \in \mathcal{RV}_{-\alpha}(\infty) \Leftrightarrow \forall\, s > 0,\; \lim_{x \to \mathbf{0}^+} \frac{f(1 - s\,x)}{f(1 - x)} = s^\alpha.$$

Part III. Tail dependence

- Definitions

→ **Transformed Archimedean copulas**

- Estimation given tail coefficients

Considered transformations

We consider transformations $T_f : [0,1] \to [0,1]$ such that

$$T_f(u) = \begin{cases} 0 & \text{if } u = 0, \\ G \circ f \circ G^{-1}(u) & \text{if } 0 < u < 1, \\ 1 & \text{if } u = 1, \end{cases} \quad (1)$$

- The transformation T_f have support $[0,1]$.
- The function G is a cdf that aims at transferring this support on the whole real line $\mathbb{R}$.
- The *conversion function* f is any continuous bijective increasing function, $f : \mathbb{R} \to \mathbb{R}$, without bounding constraints. cf Bienvenüe and R..

Considered transformations - Archimedean case

Assumption (Considered transformed generators)

Consider an initial Archimedean copula C_0 with generator ϕ_0, and the associated transformed one, $\widetilde{C}$, with generator

$$\widetilde{\phi} = T_f \circ \phi_0 \,,$$

i.e. on $(0, \infty)$,

$$\widetilde{\phi} = G \circ f \circ G^{-1} \circ \phi_0 \,.$$

One assumes that generators ϕ_0 and $\widetilde{\phi}$ satisfy admissibility conditions.

The distorted generator will depend on properties of f, ϕ_0 and G.

Regular generators

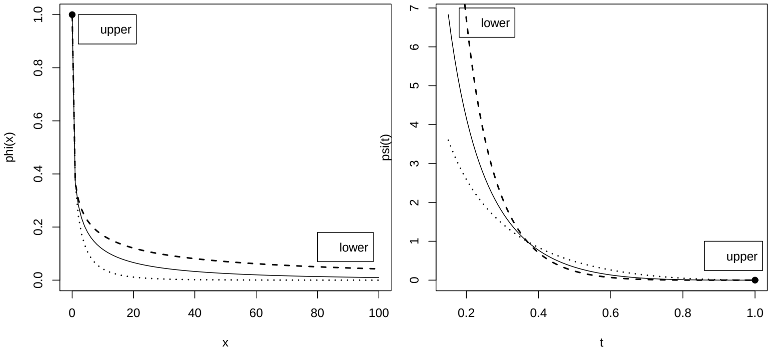


Figure : Generators $\phi^{\mathrm{Gumbel}(\theta)} = \exp(-x^{1/\theta})$ (left) and its inverse $\psi^{\mathrm{Gumbel}(\theta)} = (-\ln t)^{\theta}$ (right) for a Gumbel copula with parameters $\theta = 4$ (dashed lines), $\theta = 3$ (full lines) and $\theta = 2$ (dotted lines).

Lower tail assumptions

Assumption (Lower-tails: assumptions on f, ϕ_0, G)

Assume that f, ϕ_0 and G are continuous and differentiable functions, strictly monotone with respective proper inverse functions denoted f^{-1}, $\psi_0 = \phi_0^{-1}$ and G^{-1}. Furthermore,

i) *The function f* ***has a left asymptote*** *$\overline{f}(x) = ax + b$ as x tends to $-\infty$, for $a \in (0, +\infty)$ and $b \in (-\infty, +\infty)$.*

ii) *The inverse initial generator ψ_0* ***is regularly varying at 0*** *with some index $-r_0$, that is $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 \in [0, +\infty]$.*

iii) *The function G is a non-defective continuous c.d.f. with support $\mathbb{R}$. The following* ***rate of G is regularly varying*** *with some index $g - 1$: $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$, with $g \in (0, +\infty)$.*

Upper tail assumptions

Assumption (Upper-tails: assumptions on f, ϕ_0, G)

Assume that f, ϕ_0 and G are continuous and differentiable functions, strictly monotone with respective proper inverse functions denoted f^{-1}, $\psi_0 = \phi_0^{-1}$ and G^{-1}. Furthermore,

i) *The function f **has a right asymptote** $\bar{f}(x) = \alpha x + \beta$ as x tends to $+\infty$, for $\alpha \in (0, +\infty)$ and $\beta \in (-\infty, +\infty)$.*

ii) *The inverse initial generator ψ_0 **is regularly varying at 1** with some index ρ_0, i.e., $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$, with $\rho_0 \in [1, +\infty]$.*

iii) *The function G is a non-defective continuous c.d.f. with support $\mathbb{R}$. **The hazard rate of G is regularly varying** with some index $\gamma - 1$, that is $\mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty)$, with $\bar{G} = 1 - G$ and $\gamma \in (0, +\infty)$.*

TDC of distorted generators

Theorem (Multivariate lower TDC of transformed Archimedean copula)

If f has a left asymptote $ax+b$, $a \in (0,+\infty)$, $b \in (-\infty,+\infty)$, $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, $r_0 \in [0,+\infty]$, and the hazard rate $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$, for $g \in (0,+\infty)$. Then, transformed lower TDC is

$$\tilde{\lambda}_L^{(h,d-h)} = \begin{cases} \text{see in two slides}, & \text{if } r_0 = 0, \\ d^{-a^g r_0^{-1}} (d-h)^{a^g r_0^{-1}}, & \text{if } r_0 \in (0,+\infty), \\ 1, & \text{if } r_0 = +\infty. \end{cases} \quad (2)$$

TDC of distorted generators

Theorem (Multivariate Upper TDC of transformed Archimedean copula)

If f has a right asymptote $\alpha\, x + \beta$, $\alpha \in (0, +\infty)$, $\beta \in (-\infty, +\infty)$, $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$, $\rho_0 \in [1, +\infty]$, *and the hazard rate* $\mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty)$, $\gamma \in (0, +\infty)$. *Then, when* $\widetilde{\rho} = \rho_0\, \alpha^{-\gamma} \neq 1$, *transformed upper TDC is*

$$\widetilde{\lambda}_U^{(h,d-h)} = \begin{cases} \text{see next slide,} & \text{if } \rho_0 = 1, \\ \dfrac{\sum_{i=1}^{d} C_d^i (-1)^i \cdot i^{\alpha^\gamma \rho_0^{-1}}}{\sum_{i=1}^{d-h} C_{d-h}^i (-1)^i \cdot i^{\alpha^\gamma \rho_0^{-1}}}, & \text{if } \rho_0 \in (1, +\infty), \\ 1, & \text{if } \rho_0 = +\infty. \end{cases} \quad (3)$$

Asymptotic lower tail independence When $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 = 0$,

$$\widetilde{\lambda}_L^{(h,d-h)} = \lim_{u \to 0^+} \widetilde{\lambda}_L^{(h,d-h)}(u) = 0$$

- if $\mu_{\phi_0} = \phi_0'/\phi_0 \in \mathcal{RV}_{k_0 - 1}(\infty)$, with $k_0 \in [0, +\infty)$, if $T_f \in \mathcal{RV}_{\tilde{a}}(0)$ for $\tilde{a} \in (0, +\infty)$,

$$\widetilde{\lambda}_L^{(h,d-h)}(u) \in \mathcal{RV}_{\widetilde{z}}(0) \text{ with } \widetilde{z} = d^{k_0} - (d-h)^{k_0}.$$

Asymptotic upper tail independence When $\tilde{\psi} \in \mathcal{RV}_{\tilde{\rho}}(1)$, with $\tilde{\rho} = 1$ and if $\widetilde{\phi}$ is a d times continuously differentiable generator.

$$\widetilde{\lambda}_U^{(h,d-h)} = \lim_{u \to 1^-} \widetilde{\lambda}_U^{(h,d-h)}(u) = 0.$$

- if $(-D)^d \tilde{\phi}(0)$ is finite and not zero, where D is the derivative operator,

$$\widetilde{\lambda}_U^{(h,d-h)}(u) \in \mathcal{RV}_h(1);$$

- if $\tilde{\psi}'(1) = 0$ and the function $\widetilde{L}(s) := s \frac{\mathrm{d}}{\mathrm{d}s}\{\frac{\tilde{\psi}(1-s)}{s}\}$ is positive and $\widetilde{L} \in \mathcal{RV}_0(0)$,

$$\widetilde{\lambda}_U^{(h,d-h)}(u) \in \mathcal{RV}_0(1).$$

Part III. Tail dependence

- Definitions
- Transformed Archimedean copulas

→ **Estimation given tail coefficients**

Fit with given TDC

In the bivariate case, when $d = 2$,

$$\widetilde{\lambda}_L^{(1,1)} = 2^{-a^g r_0^{-1}} \text{ and } \widetilde{\lambda}_U^{(1,1)} = 2 - 2^{\alpha^\gamma \cdot \rho_0^{-1}}$$

Take a distortion $T_f(x) = G \circ f \circ G^{-1}(x)$ with $f = H_{m,\,h,\,p_1,\,p_2,\,\eta}$,

$$H_{m,\,h,\,p_1,\,p_2,\,\eta}(x) = m - h + (e^{p_1} + e^{p_2})\frac{x-m-h}{2} - (e^{p_1} - e^{p_2})\sqrt{\left(\frac{x-m-h}{2}\right)^2 + e^{\eta - \frac{p_1+p_2}{2}}},$$

One can deduce two parameters

$$p_1 = \frac{1}{g}\ln\left(-r_0\,\frac{\ln \widetilde{\lambda}_L^{(1,1)}}{\ln 2}\right) \quad \text{and} \quad p_2 = \frac{1}{\gamma}\ln\left(\rho_0\,\frac{\ln(2-\widetilde{\lambda}_U^{(1,1)})}{\ln 2}\right). \qquad (4)$$

In the multivariate case, when $d > 2$, same principle, but expression of p_2 more difficult to write.

distorted copulas with chosen TDC

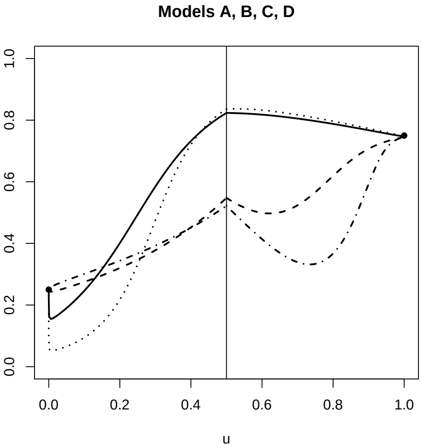


Figure : *Concentration* function $\lambda_{LU}(u) = \mathbf{1}_{\{u \le 1/2\}} \lambda_L(u) + \mathbf{1}_{\{u > 1/2\}} \lambda_U(u)$ for some distorted copulas. Chosen values: $\lambda_L = 1/4$ and $\lambda_U = 3/4$.

Conclusion

Conclusion

On the non-parametric estimation of the generator:

- Comparable performance with other estimators,
- Without solving non-linear systems of equations,
- Theoretical confidence bands.
- Class of estimators that depend on the initial copula.

On the parametric estimation of the generator:

- Analytical expressions for the level curves,
- Tunable number of parameters,
- Tail dependence can be chosen

Among further perspectives:

- Properties of the estimator (convexity, etc.)
- Estimators using other informations
- Use with nested copulas

Thank you for your attention.

papers that were partly presented

more details on Non-parametric estimation part:

Di Bernardino, E. and Rullière, D. (2013). *On certain transformations of Archimedean copulas : Application to the non-parametric estimation of their generators*. Dependence Modeling 1(1): 1-36.

more details on Parametric estimation part:

Di Bernardino, E. and Rullière, D. (2013). *Distortions of multivariate distribution functions and associated level curves: Applications in multivariate risk theory*. Insurance: Mathematics and Economics, 53(1):190-205.

Di Bernardino, E. and Rullière, D. (2014). *Estimation of multivariate critical layers: Applications to rainfall data*. Preprint available on HAL *(open access french archive website)*.

more details on Tail dependence part:

Di Bernardino, E. and Rullière, D. (2014). *On tail dependence coefficients of transformed multivariate Archimedean copulas*. Preprint available on HAL *(open access french archive website)*.