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Part I. Transformations of Archimedean copulas
Let $F$ be a $d$–dimensional cdf with marginals $F_i$, $i = 1, \ldots, d$:

By Sklar’s theorem, there exists a copula function $C : [0, 1]^d \rightarrow [0, 1]$ that links the distribution $F$ with its margins:

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

- $C$ is a $d$–dimensional cdf on $[0, 1]^d$ with uniform marginals.
- $C$ is unique if marginals $F_i$ are continuous.
Archimedean copulas

Archimedean copulas:

\[ C(u_1, \ldots, u_d) = \phi \left( \phi^{-1}(u_1) + \ldots + \phi^{-1}(u_d) \right) \]

\( \phi : \mathbb{R}^+ \rightarrow [0, 1] \) is the generator of the Archimedean copula, 
\( \phi^{-1}(x) = \inf \{ s \in \mathbb{R}^+ : \phi(s) \leq x \} \) its (generalized) inverse function.

**Generator:** \( \phi \) is continuous, decreasing, \( d \)-monotone (cf. McNeil and Nešlehová, 2009), \( \phi(0) = 1, \lim_{x \to +\infty} \phi(x) = 0. \)

**One limitation:** we consider only here strict generators, i.e. \( \forall x \in \mathbb{R}^+, \phi(x) > 0. \Rightarrow \phi \) is strictly decreasing and \( \phi^{-1} \) is the regular inverse of \( \phi. \)

**Proposition (Equivalent generators, cf. Nelsen)**

*Generator* \( \phi_a(x) = \phi(ax) \) and \( \phi(x) \) lead to the same copula, \( a \in \mathbb{R} \setminus \{0\} \)

implies that one can ask \( \phi \) to be such that \( \phi(t_0) = \varphi_0 \) for an arbitrary point \( (t_0, \varphi_0) \in \mathbb{R} \times (0, 1). \)
Proposition (Transformed copula)

Copula $\tilde{C}$ depends on a continuous increasing function $T : [0, 1] \rightarrow [0, 1]$,

$$\tilde{C}(u_1, \ldots, u_d) = T(C_0(T^{-1}(u_1), \ldots, T^{-1}(u_d))).$$

and if the initial $C_0$ is a given Archimedean copula with generator $\phi_0$, then $\tilde{C}$ is an Archimedean copula with generator

$$\tilde{\phi}(x) = T \circ \phi_0(x)$$

Admissibility conditions for $\tilde{\phi}$ (d-monotonicity, cf. McNeil and Nešlehová, 2009) ⇒ more admissibility conditions for $T$ (e.g. using Faa Di Bruno formula).

Literature on these transformations: Durrleman and al. (2000), Charpentier (2008), Valdez and Xiao (2011)
Possibility to transform both copulas and margins:

$$\tilde{F}(x_1, \ldots, x_d) = \tilde{C}(\tilde{F}_1(x_1), \ldots, \tilde{F}_d(x_d)),$$

where $\tilde{F}_i = T \circ T_i^{-1} \circ F_i$ and $T_i : [0, 1] \rightarrow [0, 1]$ are continuous and increasing.

**Why using transformations** (instead of parametric multivariate distributions)?

1. **Flexible:** allows distortions composition
   - huge variety of reachable distributions (multimodal, etc.)
   - possibility to improve a fit gradually

2. **Invertible:** analytical expressions
   - for the expression of the distribution function;
   - but also for the expression of level curves;

3. **Estimation facilities**
Part II. Estimation of the distorted Archimedean copula

→ Self-nested diagonals
  - Non-parametric estimation
  - Parametric estimation
Idea: building a non-parametric estimator of the generator $\phi$ based on the diagonal section of the copula.

**Definition (Diagonal section of the copula)**

Consider a copula $C$ satisfying *regular conditions*. For all $u \in [0, 1]$,  

$$\delta_1(u) = C(u, \ldots, u),$$

and $\delta_1^{-1}$ is the inverse function of $\delta_1$ so that $\delta_1 \circ \delta_1^{-1} = I_d$.

Remark:

- The copula $C$ is not *always* uniquely determined by its diagonal (cf. Frank’s condition, *Erdely and al. 2013*).
- Estimation based *only* on this diagonal may fail to capture tail dependence when $\phi'(0) = -\infty$. 

Definitions

Definition (Discrete self-nested diagonals)

Consider a copula $C$ satisfying regular conditions. The discrete self-nested diagonal of $C$ at order $k$ is the function $\delta_k$ such that for all $u \in [0, 1]$, $k \in \mathbb{N}$

$$
\begin{align*}
\delta_k(u) &= \delta_1 \circ \ldots \circ \delta_1(u), \quad (k \text{ times}), \\
\delta_{-k}(u) &= \delta_{-1} \circ \ldots \circ \delta_{-1}(u), \quad (k \text{ times}), \\
\delta_0(u) &= u,
\end{align*}
$$

where $\delta_1(u) = C(u, \ldots, u)$ and $\delta_{-1}$ is the inverse function of $\delta_1$, so that $\delta_1 \circ \delta_{-1}$ is the identity function.

Definition (Self-nested diagonals)

Functions of a family $\{\delta_r\}_{r \in \mathbb{R}}$ are called (extended) self-nested diagonals of a copula $C$, if $\delta_k(u)$ is the discrete self-nested diagonal of $C$ at order $k$, for all $k \in \mathbb{Z}$, and if furthermore

$$
\delta_{r_1 + r_2}(u) = \delta_{r_1} \circ \delta_{r_2}(u), \quad \forall \ r_1, r_2 \in \mathbb{R}, \forall \ u \in [0, 1].
$$
Proposition (Self-nested diagonals of an Archimedean copula)

If \( C \) is an Archimedean copula associated with a generator \( \phi \), then the self-nested diagonal of \( C \) at order \( r \) is

\[
\delta_r(x) = \phi(d^r \cdot \phi^{-1}(x)), \quad r \in \mathbb{R}.
\]

Proposition (Interpolation of self-nested diagonals)

Let \( C \) be an Archimedean copula with generator \( \phi \). For any real \( r \in [k, k + 1] \), \( k \in \mathbb{Z} \), the self-nested diagonals of \( C \) satisfies:

\[
\delta_r(x) = \phi \left( (\phi^{-1} \circ \delta_k(x))^{1-\alpha} (\phi^{-1} \circ \delta_{k+1}(x))^\alpha \right), \quad x \in [0, 1],
\]

with \( \alpha = r - \lfloor r \rfloor \) and \( k = \lfloor r \rfloor \), where \( \lfloor r \rfloor \) denotes the integer part of \( r \).

Interpolation does not depend on the Gumbel parameter in the Gumbel case.
Expression of distortions using self-nested diagonals

Proposition (Distortion $T$ using self-nested diagonals)

Consider Archimedean copulas $C_0$ and $\tilde{C}$ satisfying regular conditions and the associated self-nested diagonals $\delta_r$ and $\tilde{\delta}_r$, $r \in \mathbb{R}$. If $T$ is defined by $T(0) = 0$, $T(1) = 1$ and for all $x \in (0, 1)$,

$$T(x) = \tilde{\delta}_{r(x)}(y_0),$$

with $r(x)$ such that $\delta_{r(x)}(x_0) = x$,

then the distorted copula using distortion $T$ is equal to $\tilde{C}$: for all $u_1, \ldots, u_d$,

$$\tilde{C}(u_1, \ldots, u_d) = T \circ C_0(T^{-1}(u_1), \ldots, T^{-1}(u_d)),$$

where $(x_0, y_0) \in (0, 1)^2$ can be arbitrarily chosen. In the case where $C_0$ is the independence copula,

$$r(x) = \frac{1}{\ln d} \ln \left( \frac{-\ln x}{-\ln x_0} \right),$$
Proposition (Generator $\tilde{\phi}$ using self-nested diagonals)

Consider an Archimedean copula $\tilde{C}$ satisfying regular conditions, and the associated self-nested copulas $\tilde{\delta}_r$, for $r \in \mathbb{R}$. Assume that the copula $\tilde{C}$ is reachable by distorting an Archimedean copula $C_0$, and denote by $\delta_r$, $r \in \mathbb{R}$, the self-nested diagonals of $C_0$ and by $\phi_0$ its generator. A generator $\tilde{\phi}$ of $\tilde{C}$ is defined for all $t \in \mathbb{R}^{*+}$ by

$$\tilde{\phi}(t) = \tilde{\delta}_{\rho(t)}(y_0),$$

with $\rho(t)$ such that $\delta_{\rho(t)}(x_0) = \phi_0(t)$,

where $(x_0, y_0) \in (0, 1)^2$ can be arbitrarily chosen. In the particular case where $C_0$ is the independent copula, then

$$\rho(t) = \frac{1}{\ln d} \ln \left( \frac{t}{-\ln x_0} \right)$$
Part II. Estimation of the distorted Archimedean copula

- Self-nested diagonals
- Non-parametric estimation
- Parametric estimation
First, build a (smooth) estimator $\hat{\delta}_1$ of the diagonal of the target distorted copula $\tilde{C}$ (e.g. empirical copula, Deheuvels (1979), Fermanian and al. (2004), Omelka and al. (2009)), and denote its inverse function $\hat{\delta}^{-1}$.

Estimators of discrete self-nested diagonal of $\tilde{C}$ at order $k$ are the function $\hat{\delta}_k$ such that for all $u \in [0, 1]$, $k \in \mathbb{N}$

$$
\begin{align*}
\hat{\delta}_k(u) &= \hat{\delta}_1 \circ \ldots \circ \hat{\delta}_1(u), \quad (k \text{ times}), \\
\hat{\delta}_{-k}(u) &= \hat{\delta}_{-1} \circ \ldots \circ \hat{\delta}_{-1}(u), \quad (k \text{ times}), \\
\hat{\delta}_0(u) &= u,
\end{align*}
$$
Define self-nested diagonals at non-integer orders with a given interpolation function \( z \),

\[
\hat{\delta}_r(x) = z \left( \left( z^{-1} \circ \hat{\delta}_k(x) \right)^{1-\alpha} \left( z^{-1} \circ \hat{\delta}_{k+1}(x) \right)^{\alpha} \right), \quad x \in [0, 1],
\]

with \( \alpha = r - \lfloor r \rfloor \) and \( k = \lfloor r \rfloor \), where \( \lfloor r \rfloor \) denotes the integer part of \( r \).

Perfect interpolation functions \( z \) are functions such that for all \( r_1, r_2 \in \mathbb{R} \),

\[
\delta_{r_1} \circ \delta_{r_2} = \delta_{r_1 + r_2}.
\]

They do not depend on the Gumbel parameter in the Gumbel case (and in particular in Independent case).
Definition (Non-parametric estimation of $\tilde{\phi}$ - Case $C_0$ independent copula)

Consider an Archimedean copula $\tilde{C}$ and associated self-nested diagonals $\tilde{\delta}_r$, for $r \in \mathbb{R}$. Denote by $\hat{\delta}_r$ the estimator of $\tilde{\delta}_r$, for $r \in \mathbb{R}$. Assume that $\phi(t_0) = \varphi_0$, for a given couple of values $(t_0, \varphi_0) \in \mathbb{R}^+ \setminus \{0\} \times (0, 1)$. A non-parametric estimator $\hat{\phi}$ of $\tilde{\phi}$ is defined by $\hat{\phi}(0) = 1$ and for all $t \in \mathbb{R}^+ \setminus \{0\}$,

$$
\hat{\phi}(t) = \hat{\delta}_{\rho(t)}(\varphi_0),
$$

with $\rho(t) = \frac{1}{\ln a} \ln \left( \frac{t}{t_0} \right),$

where $(t_0, \varphi_0) \in \mathbb{R}^+ \setminus \{0\} \times (0, 1)$ can be arbitrarily chosen.

In particular, the estimator $\hat{\phi}$ of $\tilde{\phi}$ is passing through the points

$$
\{(t_k, \varphi_k)\}_{k \in \mathbb{Z}} = \{(d^k t_0, \hat{\delta}_k(\varphi_0))\}_{k \in \mathbb{Z}},
$$

No interpolation function $z$ is needed to get $(t_k, \varphi_k)$, for $k \in \mathbb{Z}$.
Non-parametric estimators of $T$ and $\tilde{\phi}$

Same kind of estimator for $\hat{T}$ or $\tilde{\phi}$ in the general case:

$$\hat{T}(x) = \delta_{r(x)}(y_0),$$
with $r(x)$ such that $\delta_{r(x)}(x_0) = x$,

where $(x_0, y_0) \in (0, 1)^2$ can be arbitrarily chosen.

Proposition (Theoretical confidence bands for estimators - not detailed here)

*Theoretical confidence bands on $\hat{\delta}_1$ $\Rightarrow$ theoretical confidence bands on $\hat{\phi}$. One needs the distribution of the empirical process $\hat{\delta}(u), u \in [0, 1]$.  

Non-parametric $\hat{\phi}(t)$

**Figure:** Estimated versus theoretical Gumbel-generator with parameter $\theta = 3$. Size of simulated samples $n = 150$ (left) and $n = 1500$ (right). Estimated $\hat{\phi}(t) = \hat{\delta}_{\rho(t)}(y_0)$ (full line). The theoretical standardized Gumbel-generator, i.e., $\tilde{\phi}(t) = \exp(-t^{1/\theta})$, is drawn using a dashed line. We force the generators to pass through the point $(t_0, \varphi_0) = (1, e^{-1})$ (black point).
Comparison with other estimators

Estimation of $\lambda$ function, $\lambda = \phi^{-1} \cdot (\phi' \circ \phi^{-1})$. Black: theoretical one. Dark green dashed line: estimator proposed by Genest and al. (2011), Pink and dotted lines: our estimator with two differentiation techniques.
Confidence bands for $\hat{\delta}_1$ imply confidence bands for $\hat{\phi}$

Figure: (Left) Confidence bands for $\hat{\delta}_1$ and $\hat{\delta}_{-1}$ for chosen parameters $\alpha^- = \beta^- = 1.05$, $\alpha^+ = \beta^+ = 0.9$. (Right) Resulting theoretical confidence band for $\hat{\phi}$. Here $\hat{C}$ is a Gumbel copula of parameter $\theta = 2$, the size of generated sample is $n = 2000$. 
Part II. Estimation of the distorted Archimedean copula

- Self-nested diagonals
- Non-parametric estimation
  → Parametric estimation
A class of parametric transformation

We take back from Bienvenüe and R. (2012):

**Definition (Conversion and distortion functions)**

Let \( f \) any bijective increasing function from \( \mathbb{R} \) to \( \mathbb{R} \). It is said to be a conversion function. The distortion \( T_f : [0, 1] \rightarrow [0, 1] \) is defined as

\[
T_f(u) = \begin{cases} 
0 & \text{if } u = 0, \\
\logit^{-1}(f(\logit(u))) & \text{if } 0 < u < 1, \\
1 & \text{if } u = 1.
\end{cases}
\]

**Remark:** Distortions function are chosen in a way to be easily invertible \((T_f \circ T_g = T_{f \circ g}, T_f^{-1} = T_{f^{-1}})\).

- We will use (composited) hyperbolic conversion functions:

\[
f(x) = H_{m, h, \rho_1, \rho_2, \eta}(x) = m - h + (e^{\rho_1} + e^{\rho_2}) \frac{x - m - h}{2} - (e^{\rho_1} - e^{\rho_2}) \sqrt{\left(\frac{x - m - h}{2}\right)^2 + e^{-\frac{\rho_1 + \rho_2}{2}}} 
\]

with \( m, h, \rho_1, \rho_2 \in \mathbb{R} \), and one smoothing parameter \( \eta \in \mathbb{R} \).

- \( H^{-1}_{m, h, \rho_1, \rho_2, \eta}(x) = H_{m, -h, -\rho_1, -\rho_2, \eta}(x) \).
Parametric distortions estimation

Framework:
- An initial given copula $C_0$ is distorted using a distortion $T$,
- Initial given margins are distorted using distortions $T_1, \ldots, T_d$.

Estimation:
- Get non-parametric estimators for $T$,
- Get non-parametric estimators for $T_1, \ldots, T_d$,
- Fit all distortions $T, T_1, \ldots, T_d$ by piecewise-linear functions (e.g. in the logit scale),
Geyser data: distorted bivariate density

Non-parametric $\Rightarrow$ Parametric estimation (without optimization).

Data: 272 eruptions of the Old Faithful geyser in Yellowstone National Park. Each observation consists of two measurements: the duration (in min) of the eruption ($X$), and the waiting time (in min) before the next eruption ($Y$).

Figure: Level curves of distorted density $\tilde{f}(x_1, x_2)$ and Old Faithful geyser data (red points).
Part III. Tail dependence

→ Definitions
  • Transformed Archimedean copulas
  • Estimation given tail coefficients
Assume that the considered copula $C$ is the distribution of some random vector $U := (U_1, \ldots, U_d)$. Denote $I = \{1, \ldots, d\}$ and consider two non-empty subsets $I_h \subset I$ and $\bar{I}_h = I \setminus I_h$ of respective cardinal $h \geq 1$ and $d - h \geq 1$. A multivariate version of classical bivariate tail dependence coefficients is given by De Luca and Riviecio, 2012 (when limits exist):

$$
\lambda^{I_h, \bar{I}_h}_L = \lim_{u \to 0^+} \lambda^{I_h, \bar{I}_h}_L(u) \quad \text{with} \quad \lambda^{I_h, \bar{I}_h}_L(u) = \mathbb{P} \left[ U_i \leq u, i \in I_h \mid U_i \leq u, i \in \bar{I}_h \right],
$$

$$
\lambda^{I_h, \bar{I}_h}_U = \lim_{u \to 1^-} \lambda^{I_h, \bar{I}_h}_U(u) \quad \text{with} \quad \lambda^{I_h, \bar{I}_h}_U(u) = \mathbb{P} \left[ U_i \geq u, i \in I_h \mid U_i \geq u, i \in \bar{I}_h \right].
$$

If for all $I_h \subset I$, $\lambda^{I_h, \bar{I}_h}_L = 0$, (resp. $\lambda^{I_h, \bar{I}_h}_U = 0$) then we say $U$ is lower tail independent (resp. upper tail independent).
With exchangeable r.v., depends on \( d = \text{card}(I) \) and \( h = \text{card}(I_h) \).

In particular case of Archimedean copulas, one have \((\psi = \phi^{-1})\):

**Proposition (Multivariate Tail Dep. Coeffs. for Archimedean copulas)**

*For Archimedean copulas the multivariate lower and upper tail dependence coefficients are respectively (cf. De Luca and Riviecio, 2012):*

\[
\lambda_L^{(h,d-h)} = \lim_{u \rightarrow 0^+} \frac{\psi^{-1} (d \psi(u))}{\psi^{-1} ((d - h) \psi(u))},
\]

\[
\lambda_U^{(h,d-h)} = \lim_{u \rightarrow 1^-} \frac{\sum_{i=0}^{d} (-1)^i C_d^i \psi^{-1} (i \psi(u))}{\sum_{i=0}^{d-h} (-1)^i C_{d-h}^i \psi^{-1} (i \psi(u))}.
\]

Figure: Shape of the \textit{concentration} function

\[ \lambda_{LU}(u) = 1_{\{u \leq 1/2\}} \lambda_L(u) + 1_{\{u > 1/2\}} \lambda_U(u) \]

for some Clayton copulas (left panel), Gumbel copulas (right panel).
At infinity:

\[ f \in RV_\alpha(\infty) \iff \forall s > 0, \lim_{x \to +\infty} \frac{f(sx)}{f(x)} = s^\alpha. \]

At zero, using \( M(x) = 1 - x \) and \( I(x) = 1/x \),

\[ f \in RV_\alpha(0) \iff f \circ I \in RV_{-\alpha}(\infty) \iff \forall s > 0, \lim_{x \to 0^+} \frac{f(sx)}{f(x)} = s^\alpha. \]

At one,

\[ f \in RV_\alpha(1) \iff f \circ M \circ I \in RV_{-\alpha}(\infty) \iff \forall s > 0, \lim_{x \to 0^+} \frac{f(1-sx)}{f(1-x)} = s^\alpha. \]
Part III. Tail dependence

- Definitions
  - Transformed Archimedean copulas
  - Estimation given tail coefficients
Considered transformations

We consider transformations \( T_f : [0, 1] \rightarrow [0, 1] \) such that

\[
T_f(u) = \begin{cases} 
0 & \text{if } u = 0, \\
G \circ f \circ G^{-1}(u) & \text{if } 0 < u < 1, \\
1 & \text{if } u = 1,
\end{cases}
\]

The transformation \( T_f \) have support \([0, 1]\).

The function \( G \) is a cdf that aims at transferring this support on the whole real line \( \mathbb{R} \).

The conversion function \( f \) is any continuous bijective increasing function, \( f : \mathbb{R} \rightarrow \mathbb{R} \), without bounding constraints. cf Bienvenüe and R. CNAM seminar, february 2015
Considered transformations - Archimedean case

Assumption (Considered transformed generators)

Consider an initial Archimedean copula $C_0$ with generator $\phi_0$, and the associated transformed one, $\tilde{C}$, with generator

$$\tilde{\phi} = T_f \circ \phi_0 ,$$

i.e. on $(0, \infty)$,

$$\tilde{\phi} = G \circ f \circ G^{-1} \circ \phi_0 .$$

One assumes that generators $\phi_0$ and $\tilde{\phi}$ satisfy admissibility conditions.

The distorted generator will depend on properties of $f$, $\phi_0$ and $G$. 

Definitions:

- **Transformed Archimedean copulas**
- **Estimation in Archimedean case**
- **Tail dependence**
- **Conclusion**

Estimation with given tail coefficients
Figure: Generators $\phi_{\text{Gumbel}}(\theta) = \exp(-x^{1/\theta})$ (left) and its inverse $\psi_{\text{Gumbel}}(\theta) = (-\ln t)^\theta$ (right) for a Gumbel copula with parameters $\theta = 4$ (dashed lines), $\theta = 3$ (full lines) and $\theta = 2$ (dotted lines).
Assumption (Lower-tails: assumptions on \( f, \phi_0, G \))

Assume that \( f \), \( \phi_0 \) and \( G \) are continuous and differentiable functions, strictly monotone with respective proper inverse functions denoted \( f^{-1} \), \( \psi_0 = \phi_0^{-1} \) and \( G^{-1} \). Furthermore,

i) The function \( f \) **has a left asymptote** \( \bar{f}(x) = ax + b \) as \( x \) tends to \(-\infty\), for \( a \in (0, +\infty) \) and \( b \in (-\infty, +\infty) \).

ii) The inverse initial generator \( \psi_0 \) is regularly varying at \( 0 \) with some index \(-r_0\), that is \( \psi_0 \in RV_{-r_0}(0) \), with \( r_0 \in [0, +\infty] \).

iii) The function \( G \) is a non-defective continuous c.d.f. with support \( \mathbb{R} \). The following **rate of \( G \) is regularly varying** with some index \( g - 1 \): \( m_G = G'/G \in RV_{g-1}(-\infty) \), with \( g \in (0, +\infty) \).
Assumption (Upper-tails: assumptions on $f$, $\phi_0$, $G$)

Assume that $f$, $\phi_0$ and $G$ are continuous and differentiable functions, strictly monotone with respective proper inverse functions denoted $f^{-1}$, $\psi_0 = \phi_0^{-1}$ and $G^{-1}$. Furthermore,

i) **The function $f$ has a right asymptote** $\bar{f}(x) = \alpha x + \beta$ as $x$ tends to $+\infty$, for $\alpha \in (0, +\infty)$ and $\beta \in (-\infty, +\infty)$.

ii) **The inverse initial generator $\psi_0$ is regularly varying at 1** with some index $\rho_0$, i.e., $\psi_0 \in RV_{\rho_0}(1)$, with $\rho_0 \in [1, +\infty]$.

iii) **The function $G$ is a non-defective continuous c.d.f. with support $\mathbb{R}$. The hazard rate of $G$ is regularly varying** with some index $\gamma - 1$, that is $\mu_G = G'/\bar{G} \in RV_{\gamma-1}(\infty)$, with $\bar{G} = 1 - G$ and $\gamma \in (0, +\infty)$. 
If \( f \) has a left asymptote \( ax + b, \ a \in (0, +\infty), \ b \in (-\infty, +\infty) \), \( \psi_0 \in \mathcal{RV}_{-r_0}(0), \ r_0 \in [0, +\infty], \) and the hazard rate \( m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty), \) for \( g \in (0, +\infty) \). Then, transformed lower TDC is

\[
\tilde{\lambda}_L^{(h, d-h)} = \begin{cases} 
\text{see in two slides}, & \text{if } r_0 = 0, \\
d^{-ag} r_0^{-1} (d - h)^a g r_0^{-1}, & \text{if } r_0 \in (0, +\infty), \\
1, & \text{if } r_0 = +\infty.
\end{cases}
\]
Theorem (Multivariate Upper TDC of transformed Archimedean copula)

If \( f \) has a right asymptote \( \alpha x + \beta, \alpha \in (0, +\infty), \beta \in (-\infty, +\infty) \),
\( \psi_0 \in \mathcal{RV}_{\rho_0}(1), \rho_0 \in [1, +\infty], \) and the hazard rate
\( \mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty), \gamma \in (0, +\infty). \) Then, when \( \tilde{\rho} = \rho_0 \alpha^{-\gamma} \neq 1, \)
transformed upper TDC is

\[
\tilde{\lambda}_U^{(h,d-h)} = \begin{cases} 
\text{see next slide,} & \text{if } \rho_0 = 1, \\
\frac{\sum_{i=1}^{d} C_i (-1)^i \cdot i^{\alpha \gamma \rho_0^{-1}}}{\sum_{i=1}^{d-h} C_i (-1)^i \cdot i^{\alpha \gamma \rho_0^{-1}}}, & \text{if } \rho_0 \in (1, +\infty), \\
1, & \text{if } \rho_0 = +\infty. 
\end{cases} 
\]
Asymptotic lower tail independence When $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 = 0$,

$$\lambda_L^{(h,d-h)} = \lim_{u \to 0^+} \lambda_L^{(h,d-h)}(u) = 0$$

- if $\mu_{\phi_0} = \phi_0'/\phi_0 \in \mathcal{RV}_{k_0-1}(\infty)$, with $k_0 \in [0, +\infty)$, if $T_f \in \mathcal{RV}_{\bar{a}}(0)$ for $\bar{a} \in (0, +\infty),

$$\lambda_L^{(h,d-h)}(u) \in \mathcal{RV}_{\bar{z}}(0) \text{ with } \bar{z} = d^{k_0} - (d - h)^{k_0}.$$

Asymptotic upper tail independence When $\tilde{\psi} \in \mathcal{RV}_{\tilde{\rho}}(1)$, with $\tilde{\rho} = 1$ and if $\tilde{\phi}$ is a $d$ times continuously differentiable generator.

$$\lambda_U^{(h,d-h)} = \lim_{u \to 1^-} \lambda_U^{(h,d-h)}(u) = 0.$$

- if $(-D)^d \tilde{\phi}(0)$ is finite and not zero, where $D$ is the derivative operator,

$$\lambda_U^{(h,d-h)}(u) \in \mathcal{RV}_{h}(1);$$

- if $\tilde{\psi}'(1) = 0$ and the function $\tilde{L}(s) := s \frac{d}{ds} \left\{ \frac{\tilde{\psi}(1-s)}{s} \right\}$ is positive and $\tilde{L} \in \mathcal{RV}_0(0)$,

$$\lambda_U^{(h,d-h)}(u) \in \mathcal{RV}_0(1).$$
Part III. Tail dependence

- Definitions
- Transformed Archimedean copulas
  → Estimation given tail coefficients
In the bivariate case, when $d = 2$, 

$$\tilde{\lambda}_L^{(1,1)} = 2^{-ag} r_0^{-1} \text{ and } \tilde{\lambda}_U^{(1,1)} = 2 - 2^{\alpha \gamma \cdot \rho_0^{-1}}$$

Take a distortion $T_f(x) = G \circ f \circ G^{-1}(x)$ with $f = H_{m, h, p_1, p_2, \eta}$,

$$H_{m, h, p_1, p_2, \eta}(x) = m - h + (e^{p_1} + e^{p_2}) \frac{x - m - h}{2} - (e^{p_1} - e^{p_2}) \sqrt{\left( \frac{x - m - h}{2} \right)^2 + e^{\eta - \frac{p_1 + p_2}{2}}}$$

One can deduce two parameters

$$p_1 = \frac{1}{g} \ln \left( -r_0 \frac{\ln \tilde{\lambda}_L^{(1,1)}}{\ln 2} \right) \text{ and } p_2 = \frac{1}{\gamma} \ln \left( \rho_0 \frac{\ln (2 - \tilde{\lambda}_U^{(1,1)})}{\ln 2} \right).$$ \hspace{1cm} \text{(4)}$$

In the multivariate case, when $d > 2$, same principle, but expression of $p_2$ more difficult to write.
distorted copulas with chosen TDC

 Models A, B, C, D

\[ \lambda_{LU}(u) = 1_{\{u \leq 1/2\}} \lambda_L(u) + 1_{\{u > 1/2\}} \lambda_U(u) \]

for some distorted copulas. Chosen values: \( \lambda_L = 1/4 \) and \( \lambda_U = 3/4 \).

Figure: *Concentration* function \( \lambda_{LU}(u) \) for some distorted copulas. Chosen values: \( \lambda_L = 1/4 \) and \( \lambda_U = 3/4 \).
Conclusion
Conclusion

On the non-parametric estimation of the generator:
- Comparable performance with other estimators,
- Without solving non-linear systems of equations,
- Theoretical confidence bands.
- Class of estimators that depend on the initial copula.

On the parametric estimation of the generator:
- Analytical expressions for the level curves,
- Tunable number of parameters,
- Tail dependence can be chosen

Among further perspectives:
- Properties of the estimator (convexity, etc.)
- Estimators using other informations
- Use with nested copulas
Thank you for your attention.
papers that were partly presented

more details on **Non-parametric estimation** part:


more details on **Parametric estimation** part:


more details on **Tail dependence** part: