MULTIVARIATE EXTREME VALUE ANALYSIS UNDER A DIRECTIONAL APPROACH

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(A) Classical direction \( u \)

(B) Auxiliary direction \( u \)
Multivariate framework and directions

Same view, different perspectives
The lack of a total order in high dimensions.
Drawbacks in the multivariate setting

- The lack of a total order in high dimensions.
- The dependence among the variables belonging to a system.
Drawbacks in the Multivariate Setting

- The lack of a total order in high dimensions.
- The dependence among the variables belonging to a system.
- There are many interesting directions to analyze the data.
Drawbacks in the Multivariate Setting

- The lack of a total order in high dimensions.
- The dependence among the variables belonging to a system.
- There are many interesting directions to analyze the data.
- The costs of computing in high dimensions.
Introduce a directional multivariate setting for extreme value analysis
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1. Considering the dependence among the variables.
2. Giving the possibility of analyzing the variables considering external information, manager preferences or intrinsic system characteristics.
3. Improving the interpretation of the analysis of extremes.
Introduce a directional multivariate setting for extreme value analysis

1. Considering the dependence among the variables.
2. Giving the possibility of analyzing the variables considering external information, manager preferences or intrinsic system characteristics.
3. Improving the interpretation of the analysis of extremes.
4. Providing a non-parametric procedure for estimation to compute the analysis in high dimensions.
1 Directional Basic Concepts

2 Non-Parametric Estimation

3 Applications in Extreme Value Analysis
   - Environmental Applications
   - Application in Financial Risk Measures
   - Current Research about Estimation

4 Conclusions and Future Research
**Definition**

Given $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$ and $||\mathbf{u}|| = 1$, the orthant with vertex $\mathbf{x}$ and direction $\mathbf{u}$ is:

$$\mathcal{O}_x^u = \{ \mathbf{z} \in \mathbb{R}^n | R_u (\mathbf{z} - \mathbf{x}) \geq 0 \},$$

where $\mathbf{e} = \frac{1}{\sqrt{n}}(1, \ldots, 1)'$ and $R_u$ is a matrix such that $R_u \mathbf{u} = \mathbf{e}$. 

\[
\begin{align*}
C^u_x & \equiv \text{Oriented Orthant.}
\end{align*}
\]
Examples of Oriented Orthants

(A) Orthant in direction $\mathbf{u} = (0, 1)$  
(B) Orthant in direction $\mathbf{u} = -\mathbf{e}$

Examples of oriented orthants in $\mathbb{R}^2$
Given \( u \in \mathbb{R}^n, \|u\| = 1 \) and a random vector \( X \) with distribution probability \( P \), the \( \alpha \)-quantile curve in direction \( u \) is defined as:

\[
Q_X(\alpha, u) := \partial \{ x \in \mathbb{R}^n : P [C_x^u] \leq \alpha \},
\]

where \( \partial \) mans the boundary and \( 0 \leq \alpha \leq 1 \).
**Definition**

Those sets are defined by:

\[ \mathcal{U}_X(\alpha, u) := \{ x \in \mathbb{R}^n : \mathbb{P}[C_x^u] < \alpha \}, \]

\[ \mathcal{L}_X(\alpha, u) := \{ x \in \mathbb{R}^n : \mathbb{P}[C_x^u] > \alpha \}. \]
\( u \in \mathcal{U} = \left\{ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\} \)

(A) Bivariate Uniform  (B) Bivariate Exponential  (C) Bivariate Normal

CLASSICAL DIRECTIONS
Directional Basic Concepts

**Directional Multivariate Level-Sets**

\[ \mathbf{u} \in \mathcal{U} = \{(1, 0), (0, 1), (-1, 0), (0, -1)\} \]

(A) Bivariate Uniform  (B) Bivariate Exponential  (C) Bivariate Normal

**Canonical Directions**
\( X_m := \{x_1, \cdots, x_m\} \), the sample data of the random vector \( X \),

\( \mathbb{P}_{X_m}[\cdot] \) is the empirical probability law of \( X_m \),

\( \hat{Q}^h_{X_m}(\alpha, u) := \left\{ x_j : |\mathbb{P}_{X_m}[\mathcal{C}^u_{x_j}] - \alpha| \leq h \right\} \) the sample quantile curve with a slack \( h \), avoiding an empty set of estimated quantiles.

\( \hat{U}^h_{X_m}(\alpha, u) := \left\{ x_j : \mathbb{P}_{X_m}[\mathcal{C}^u_{x_j}] < \alpha - h \right\} \) the sample upper \( \alpha \)-level set with a slack \( h \),

\( \hat{L}^h_{X_m}(\alpha, u) := \left\{ x_j : \mathbb{P}_{X_m}[\mathcal{C}^u_{x_j}] > \alpha + h \right\} \) the sample lowe \( \alpha \)-level set with a slack \( h \).
Input: \( u, \alpha, h \) and the multivariate sample \( X_m \).

for \( i = 1 \) to \( m \)

\[
P_i = \mathbb{P}_{X_m} \left[ \mathcal{C}_x^{u} \right],
\]

If \( |P_i - \alpha| \leq h \)

\[
x_i \in \hat{Q}^{h}_{X_m} (\alpha, u),
\]

end

If \( P_i < \alpha - h \)

\[
x_i \in \hat{U}^{h}_{X_m} (\alpha, u),
\]

end

If \( P_i > \alpha + h \)

\[
x_i \in \hat{L}^{h}_{X_m} (\alpha, u),
\]

end
## Execution Time

### Time in Seconds

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<th>1000</th>
<th>5000</th>
<th>10000</th>
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In an Intel core i7 (3.4 GH) computer with 32 Gb RAM.
Roughly speaking, a $n$-copula $C$ is a particular type of distribution with domain in the unit hyper-cube and uniform margins.

**Sklar’s Theorem**

Let $F$ be a $n$-dimensional distribution function with marginals $F_1, \ldots, F_n$. Then there exists a $n$-copula $C$ such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

If $F_1, \ldots, F_n$ are continuous, then $C$ is unique.
Let $C$ be the $n$–copula of $X$ and $F_i$, $i = 1, \ldots, d$ its margins. Then for $0 < \alpha < 1$ the corresponding $\alpha$–quantile hyper-curve, upper and lower level sets are defined as:

\[
\begin{align*}
\{ x \in \mathbb{R}^n \text{ such that } x_i &= F_{X_i}^{-1}(v_i); \ i = 1, \ldots, n; \ v \in [0, 1]^n : C(v) = \alpha \}, \\
\{ x \in \mathbb{R}^n \text{ such that } x_i &= F_{X_i}^{-1}(v_i); \ i = 1, \ldots, n; \ v \in [0, 1]^n : C(v) < \alpha \}, \\
\{ x \in \mathbb{R}^n \text{ such that } x_i &= F_{X_i}^{-1}(v_i); \ i = 1, \ldots, n; \ v \in [0, 1]^n : C(v) > \alpha \}.
\end{align*}
\]
Applications

Environmental

EXTREMES THROUGH COPULAS (REVIEW)

Handle Copula Families & Marginal Distributions (G.E.V, etc)

Sklar’s Theorem

Closed Multivariate Quantile Expressions
This approach is hard to manipulate because:

- The parametric nature generates complications and/or restrictions in high dimension, even using nested copula techniques.
- The models are quite rigid.
Directional Multivariate Level-Sets

Directional Extremes
Why is useful a directional approach in environmental engineering?

To solve this question we simulate data from the model in Salvadori et al. (2011), which refers to data of maximum annual flood peaks $Q$, volumes $V$ and water levels $L$ in the Ceppo Morelli dam, Italy.
Original dataset of the Ceppo Morelli dam
Copula model for the Ceppo Morelli dam
Simulated Sample from the model in Salvadori et al.
Extremes with the non-parametric approach,
in the classic direction for $\alpha = 1\%$
Extremes with the non-parametric approach, in the first $PCA$ direction for $\alpha = 1\%$
Final level from the simulated occurrences of $Q, V, L$
## Measures of False-Positives and True-Positives in Classic and Directional Approaches

<table>
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<th>False-Positives</th>
<th>True-Positives</th>
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<tbody>
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<td><strong>Classic Direction</strong></td>
<td>90%</td>
<td>100%</td>
</tr>
<tr>
<td><strong>PCA Direction</strong></td>
<td>35%</td>
<td>100%</td>
</tr>
</tbody>
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Torres Díaz, Raúl A.  
Directional Extreme Value Analysis  
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**Theorem**

Let $\mathbf{u}$ be fixed, then the directional quantiles of a random vector $\mathbf{X}$ with *regularity conditions* are equivalent to those obtained by the copula procedure on the random vector $R_\mathbf{u}\mathbf{X}$, where $R_\mathbf{u}$ is the rotation matrix in the orthant definition.
Given \( \alpha \)

\[ Q_x(\alpha, -e) \equiv F_x^{-1}(\alpha) \quad \& \quad Q_x(\alpha, e) \equiv \bar{F}_x^{-1}(\alpha) \]

Under regularity conditions,

Orthogonal Quasi-Invariance \& Sklar’s Theorem over \( R_{\pm}X \)

Directional approach \( \equiv \) Copula approach.
Let $X$ be a random variable representing loss, $F$ its distribution function and $0 \leq \alpha \leq 1$. Then,

$$VaR_\alpha(X) := \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\}.$$
Let $X$ be a random variable representing loss, $F$ its distribution function and $0 \leq \alpha \leq 1$. Then,

$$\text{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\}.$$
Let $X$ be a random variable representing loss, $F$ its distribution function and $0 \leq \alpha \leq 1$. Then,

$$\text{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\}.$$
The VaR has become a benchmark for risk management.

The VaR has been criticized by Artzner et al. (1999) since it does not encourage diversification.

But defended by Heyde et al. (2009) for its robustness and recently by Danielsson et al. (2013) for its tail subadditivity.
There is not a unique definition of a multivariate quantile.
There are a lot of assets in a portfolio. (High Dimension)
There is dependence among them.
An initial idea to study risk measures related to portfolios

\[ X = (X_1, \ldots, X_n), \]

is to consider a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and then:

- The \textit{VaR} of the joint portfolio is the univariate-one associated to \( f(X) \).
An initial idea to study risk measures related to portfolios

\[ X = (X_1, \ldots, X_n), \]

is to consider a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and then:

- The VaR of the joint portfolio is the univariate-one associated to \( f(X) \).
- In Burgert and Rüschendorf (2006),

\[
    f(X) = \sum_{i=1}^{n} X_i \quad \text{or} \quad f(X) = \max_{i \leq n} X_i.
\]

Output: A NUMBER
Embrechts and Puccetti (2006) introduced a multivariate approach of the Value at Risk,
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- **Multivariate lower-orthant Value at Risk**

  \[ \overline{VaR}_\alpha(X) := \partial \{ x \in \mathbb{R}^n \mid F_X(x) \geq \alpha \}. \]

- **Multivariate upper-orthant Value at Risk**

  \[ \underline{VaR}_\alpha(X) := \partial \{ x \in \mathbb{R}^n \mid \bar{F}_X(x) \leq 1 - \alpha \}. \]

**Output:** A SURFACE ON \( \mathbb{R}^n \)
Cousin and Di Bernardino (2013) introduced a multivariate risk measure related to the measure introduced by Embrechts and Puccetti (2006).
Cousin and Di Bernardino (2013) introduced a multivariate risk measure related to the measure introduced by Embrechts and Puccetti (2006).

- Multivariate lower-orthant Value at Risk
  \[
  \text{VaR}_\alpha(X) := \mathbb{E} [X | F_X(x) = \alpha].
  \]

- Multivariate upper-orthant Value at Risk
  \[
  \overline{\text{VaR}}_\alpha(X) := \mathbb{E} [X | \bar{F}_X(x) = 1 - \alpha].
  \]

Output: A POINT IN $\mathbb{R}^n$
Let $X$ be a random vector satisfying "the regularity conditions", then the Value at Risk of $X$ in direction $u$ and confidence parameter $\alpha$ is defined as

$$VaR^u_\alpha(X) = \left(Q_X(\alpha, u) \cap \{\lambda u + \mathbb{E}[X]\}\right),$$

where $\lambda \in \mathbb{R}$ and $0 \leq \alpha \leq 1$. 

Output: A point in $\mathbb{R}^n$
**Directional Multivariate Value at Risk (MVaR)**

(A) Bivariate Uniform  (B) Bivariate Exponential  (C) Bivariate Normal

\[ \text{VaR}_{0.7}^e(X) \]
(A) Bivariate Uniform  (B) Bivariate Exponential  (C) Bivariate Normal

$$VaR^e_{0.3}(X)$$
**Non-Negative Loading**: If $\lambda > 0$,

$$\mathbb{E}[X] \preceq_u \text{VaR}_\alpha^u(X),$$

where the order is given by

**Preorder (Laniado et al. (2010))**

$x$ is said to be less than $y$ if:

$$x \preceq_u y \equiv \mathcal{C}_x^u \supseteq \mathcal{C}_y^u \equiv R_u x \leq R_u y.$$
Quasi-Odd Measure: \( \text{VaR}_\alpha^u(-X) = -\text{VaR}_\alpha^{-u}(X) \).

Positive Homogeneity and Translation Invariance: Given \( c \in \mathbb{R}^+ \) and \( b \in \mathbb{R}^n \), then

\[
\text{VaR}_\alpha^u(cX + b) = c \text{VaR}_\alpha^u(X) + b.
\]
Orthogonal Quasi-Invariance: Let \( w \) and \( Q \) be an unit vector and a particular orthogonal matrix obtained by a QR decomposition such that \( Qu = w \). Then,

\[
VaR^w_\alpha(QX) = QVaR^u_\alpha(X).
\]
**Consistency:** Let \( X \) and \( Y \) be random vectors such that \( \mathbb{E}[Y] = cu + \mathbb{E}[X] \), for \( c > 0 \) and \( X \leq_{\mathbb{E}} Y \). Then:

\[
\text{VaR}^{\mathbb{u}}_{\alpha}(X) \preceq_u \text{VaR}^{\mathbb{u}}_{\alpha}(Y),
\]

where the stochastic order is defined by

**Stochastic Extremal Order (Laniado et al. (2012))**

Let \( X \) and \( Y \) be two random vectors in \( \mathbb{R}^n \),

\[
X \leq_{\mathbb{E}} Y \equiv \mathbb{P}[R_u(X - z) \geq 0] \leq \mathbb{P}[R_u(Y - z) \geq 0] \equiv \mathbb{P}_X[\mathcal{C}^{\mathbb{u}}_z] \leq \mathbb{P}_Y[\mathcal{C}^{\mathbb{u}}_z],
\]

for all \( z \) in \( \mathbb{R}^n \).
**MVar Properties**

- **Non-Excessive Loading**: For all $\alpha \in (0, 1)$ and $u \in \mathbb{B}(0)$,
  \[
  \text{VaR}_\alpha^{u}(X) \preceq_{u} R'_{u} \sup_{\omega \in \Omega} \{R_{u}X(\omega)\}.
  \]

- **Subadditivity in the Tail Region**: Let $X$ and $Y$ be random vectors, with the same mean $\mu$ and let $(R_{u}X, R_{u}Y)$ be a regularly varying random vector. Then,
  \[
  \text{VaR}_\alpha^{u}(X + Y) \preceq_{u} \text{VaR}_\alpha^{u}(X) + \text{VaR}_\alpha^{u}(Y).
  \]
RESULT

Let $\mathbf{X}$ be a random vector and $\mathbf{u}$ a direction. Then for all $0 \leq \alpha \leq 1$,

$$\text{VaR}^u_\alpha(\mathbf{X}) \preceq_\mathbf{u} \text{VaR}^{-u}_{1-\alpha}(\mathbf{X}).$$
Then, analogously as Embrechts and Puccetti (2006) and Cousin and Di Bernardino (2013), we can define:

*Lower Multivariate VaR in the direction* $u$ *as*

$$\text{VaR}_{\alpha}^u(X),$$

*Upper Multivariate VaR in the direction* $u$ *as*

$$\text{VaR}_{1-\alpha}^{-u}(X).$$
Lower Multivariate VaR = $\text{VaR}_{0.3}^e(X)$ and
Upper Multivariate VaR = $\text{VaR}_{0.7}^{-e}(X)$
Lower Multivariate VaR = \( \text{VaR}_{0.3}^{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)} (X) \) and

Upper Multivariate VaR = \( \text{VaR}_{0.7}^{\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)} (X) \)
Let $\mathbf{X}$ be a random vector with survival function $\bar{F}$ quasi-concave. Then, for all $\alpha \in (0, 1)$:

$$VaR_{1-\alpha}(X_i) \geq [VaR^e_\alpha(\mathbf{X})]_i,$$

for all $i = 1, \ldots, n$.

Moreover, if its distribution function $F$ is quasi-concave, then, for all $\alpha \in (0, 1)$,

$$[VaR^{-e}_{1-\alpha}(\mathbf{X})]_i \geq VaR_{1-\alpha}(X_i),$$

for all $i = 1, \ldots, n$. 
RESULT

Let \( \mathbf{X} \) be a random vector and \( \mathbf{u} \) a direction. If the survival function of \( R_u \mathbf{X} \) is quasi-concave. Then, for all \( 0 \leq \alpha \leq 1 \),

\[
VaR_{1-\alpha}([R_u \mathbf{X}]_i) \geq [R_u VaR_{\alpha}^u(\mathbf{X})]_i, \quad \text{for all } i = 1, \ldots, n.
\]

And if \( R_u \mathbf{X} \) has a quasi-concavity cumulative distribution, we have that

\[
[R_u VaR_{1-\alpha}^-u(\mathbf{X})]_i \geq VaR_{1-\alpha}([R_u \mathbf{X}]_i), \quad \text{for all } i = 1, \ldots, n.
\]
We analyze the behavior of the $MVaR$ when a sample is contaminated with different types of outliers.

We use as a benchmark the measurement given by the multivariate $VaR$ in Cousin and Di Bernardino (2013).
ROBUSTNESS

We simulate 5000 observations of the following random vector:

\[
X^\omega \overset{d}{=} \begin{cases}
X_1 & \text{with probability } p = 1 - \omega, \\
X_2 & \text{with probability } p = \omega,
\end{cases}
\]

where \( X_1 \overset{d}{=} N_1(\mu_1, \Sigma_1) \), \( X_2 \overset{d}{=} N_2(\mu_1 + \Delta \mu, \Sigma_1 + \Delta \Sigma) \) and \( 0 \leq \omega \leq 1 \).

Specifically:

\[
\mu_1 = [50, 50]', \quad \Sigma_1 = \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{pmatrix}.
\]

Contaminating:

1. Varying only the mean.
2. Varying only the variances.
3. Varying all the parameters.
To evaluate the impact of the contamination, we use:

\[ PV^\omega = \frac{||\text{Measure}(X^\omega) - \text{Measure}(X^0)||_2}{||\text{Measure}(X^0)||_2}, \]

where \( \text{Measure}(X^0) \) is the sample with \( \omega = 0\% \) and \( \text{Measure}(X^\omega) \) is the sample with level of contamination \( \omega\% \), \( (\omega = 1\% \rightarrow 10\%) \).
1. Varying only the mean, \( \Delta \mu \neq 0, \quad \Delta \Sigma = 0 \).

\[
(A) \quad \Delta \mu = (20, 20)'
\]

\[
(B) \quad \Delta \mu = (0, 50)'
\]

Mean of \( PV^\omega \)
2. Varying only the variances, \( \Delta \mu = 0 \), \( \Delta \Sigma = \begin{bmatrix} 4.5 & 0 \\ 0 & 6.5 \end{bmatrix} \),
3. Varying all the parameters, $\Delta \mu \neq 0$, $\Delta \Sigma = \begin{bmatrix} 4.5 & 0.2 \\ 0.3 & 6.5 \end{bmatrix}$.

(A) $\Delta \mu = (20, 20)'$

(B) $\Delta \mu = (0, 50)'$

Mean of $PV^\omega$
Let $X$ be a multivariate regularly varying random vector with tail index $\beta$. Then for all orthogonal transformation $Q$ the random vector $QX$ is regularly varying with tail index $\beta$. 
Using as base the theory in De Haan and Huang (1995) for a bivariate estimation of quantile curves, we want to:

- Study the extension to higher dimensions.
Using as base the theory in De Haan and Huang (1995) for a bivariate estimation of quantile curves, we want to:

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Using as base the theory in De Haan and Huang (1995) for a bivariate estimation of quantile curves, we want to:

- Study the extension to higher dimensions.
- Link the directional notion into the theory by using previous result.
- Study convergence and consistency, theoretically and practically.
Using as base the theory in De Haan and Huang (1995) for a bivariate estimation of quantile curves, we want to:

- Study the extension to higher dimensions.
- Link the directional notion into the theory by using previous result.
- Study convergence and consistency, theoretically and practically.
- Make comparisons between the previous non-parametric approach and the resultant by this research.
(A) $n = 500, \quad \alpha = \frac{1}{n}$  

(B) $n = 5000, \quad \alpha = \frac{1}{n}$

**Bivariate $t$-distribution with $\nu = 3$**
Estimation in the first PCA direction

\[(A) \; n = 500, \; \alpha = \frac{1}{n} \]

\[(B) \; n = 5000, \; \alpha = \frac{1}{n} \]

Bivariate $t$-distribution with $\nu = 3$
We have introduced an extension of the multivariate extreme value analysis by introducing a directional notion.
Conclusions and Future Research

- We have introduced an extension of the multivariate extreme value analysis by introducing a directional notion.
- The directional approach allows to consider external intrinsic information of a system or management preferences in the analysis.
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The directional approach allows to consider external intrinsic information of a system or management preferences in the analysis.

We provide arguments of the needing of these directional approach in practice. As well as, we present theoretical properties and results of this directional extension in the developed applications.
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The directional approach allows to consider external intrinsic information of a system or management preferences in the analysis.

We provide arguments of the needing of these directional approach in practice. As well as, we present theoretical properties and results of this directional extension in the developed applications.

Two important aspects have been studied. Asymptotic convergence and robustness in more general cases.
Torres R., Lillo R.E., and Laniado H.,
A directional multivariate Value at Risk.
Insurance Math. Econom., 2015 Accepted.
http://dx.doi.org/10.1016/j.insmatheco.2015.09.002

Nelsen R.B.,
An Introduction to copulas, (2nd edn)

Shaked M. and Shanthikumar J.,
Stochastic Orders and their Applications.

De Haan L., and Huang X. (1995)
Large Quantile Estimation in a Multivariate Setting,

Arbia, G.,
Bivariate value at risk.

Artzner P., Delbae, F., Heath J. Eber D.,
Coherent measures of risk.
BIBLIOGRAPHY

Burgert C. and Rüschendorf L.,
*On the optimal risk allocation problem.*

Cardin M. and Pagani E.,

Cascos I. and Molchanov I.,
*Multivariate risks and depth-trimmed regions.*

Cascos I., López A. and Romo J.,
*Data depth in multivariate statistics.*

Cascos I. and Molchanov I.,
*Multivariate risk measures: a constructive approach based on selections.*

Cousin A. and Di Bernardino E.,
*On multivariate extensions of Value-at-Risk.*
Daníelsson J. et al.,
*Fat tails, VaR and subadditivity.*

Embrechts P. and Puccetti G.,
*Bounds for functions of multivariate risks.*

Fernández-Ponce J. and Suárez-Llorens A.,
*Central regions for bivariate distributions.*

Fraiman R. and Pateiro-López B.,
*Quantiles for finite and infinite dimensional data.*

Hallin M., Paindaveine D. and Šiman M.,
*Multivariate quantiles and multiple-output regression quantiles: from L1 optimization to halfspace depth.*

Heyde C., Kou S. and Peng X.,
*What is a good external risk measure: bridging the gaps between robustness, subadditivity, and insurance risk measure.*

Jessen A. and Mikosh T.,
*Regularly varying functions.*
Kong L. and Mizera I.,
*Quantile tomography: Using quantiles with multivariate data.*

Laniado H., Lillo R. and Romo J.,
*Extremality in Multivariate Statistics.*

Laniado H., Lillo R. and Romo J.,
*Multivariate extremality measure.*

Nappo G. and Spizzichino F.,
*Kendall distributions and level sets in bivariate exchangeable survival models.*
Information Sciences 179, 2878-2890, 2009.

Rachev S., Ortabelli S., Stoyanov S., Fabozzi F.,
*Desirable Properties of an Ideal Risk Measure in Portfolio Theory.*

Serfling R.,
*Quantile functions for multivariate analysis: approaches and applications.*

Zuo Y. and Serfling R.,
*General notions of statistical depth function.*
Thanks