Pseudopotentials and Loading Surfaces for an Endochronic Plasticity Theory with Isotropic Damage

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Abstract: The endochronic theory, developed in the early 70s, allows the plastic behavior of materials to be represented by introducing the notion of intrinsic time. With different viewpoints, several authors discussed the relationship between this theory and the classical theory of plasticity. Two major differences are the presence of plastic strains during unloading phases and the absence of an elastic domain. Later, the endochronic plasticity theory was modified in order to introduce the effect of damage. In the present paper, a basic endochronic model with isotropic damage is formulated starting from the postulate of strain equivalence. Unlike the previous similar analyses, in this presentation the formal tools chosen to formulate the model are those of convex analysis, often used in classical plasticity: namely pseudopotentials, indicator functions, subdifferentials, etc. As a result, the notion of *loading surface* for an endochronic model of plasticity with damage is investigated and an insightful comparison with classical models is made possible. A damage pseudopotential definition allowing a very general damage evolution is given.

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Introduction

In the early 1970s, Valanis (1971) proposed the *endochronic theory of viscoplasticity*, which postulates the existence of an *intrinsic time* governing the rate-independent evolution of stress and strains in materials, whereas the Newtonian time is exploited to model the viscous behavior; see also (Schapery 1968; Bažant and Bhat 1976). In the case of plasticity without viscous effects, the resulting constitutive laws are characterized by the absence of an elastic domain and the corresponding hysteresis loops are typically smooth and open. The flow rules of these models were not originally formulated in terms of pseudopotentials, which made the direct comparison of this class of models with classical plasticity theories difficult (Valanis 1980). However, it was recently proven by Erlicher and Point (2006) that endochronic models do admit a representation based on pseudopotentials and on the normality assumption, provided that pseudopotentials be endowed with an additional dependence on state variables. This proof, given for the case of plastically incompressible models, showed the strong relationship between the endochronic theory and the generalized plasticity (Phillips and Sierakowski 1965; Eisenberg and Phillips 1971; Lubliner et al. 1993; Auricchio and Taylor 1995). It was also shown that the nonlinear kinematic hardening model, that is associated, but is not in a generalized sense, admits a representation in terms of a pseudopotential. Recently, the same authors extended this analysis to other models, like the Mróz model (Point and Erlicher 2008) and the nonassociated Drucker–Prager model (Erlicher and Point 2005); see also Ziegler and Wehrli (1987) and Houlsby and Puzrin (2000). In summary, this thermodynamically well-posed approach can be used for a very large class of existing *classical* or *nonclassical* plasticity models. Actually, a similar approach is used in geotechnical engineering, see, e.g., Collins and Houlsby (1997), where pseudopotentials have an additional dependence on the so-called *true stresses*, distinguished from the *generalized stresses*.

The standard endochronic theory was modified by several authors through the introduction of a damage variable. Using the strain equivalence postulate, Xiaode (1989) proposed an endochronic model with isotropic damage, whereas Valanis (1990) discussed an endochronic model with anisotropic damage, in the larger theoretical framework of fracture mechanics. Later, a different approach based on the postulate of energy equivalence was used, among others, by Chow and Chen (1992) and Wu and Nanakorn (1998, 1999).

In the aforementioned works, the thermodynamic formulation of flow rules is not based on the notions of pseudopotentials and loading surfaces, as it is typical for other classical plasticity models with or without damage. Hence, in this paper, a simple endochronic model of plasticity with isotropic damage similar to that discussed by Xiaode (1989) is presented: no generalization is introduced with respect to the previously cited models, but a new approach is suggested for their description. In detail, the postulate of strain equivalence is adopted; the Helmholtz energy is assumed to have a regular quadratic term and an additional singular term; the tools of the convex analysis such as indicator functions and subdifferentials (Rockafellar 1969; Moreau 1970; Frémond 2002) are used to define the flow rules starting from well-suited pseudopotentials. This presentation leads to the proper definition of the plasticity loading surface for an endochronic

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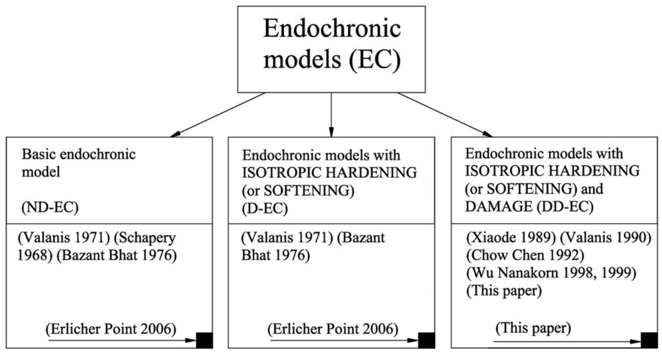

Fig. 1. Classification of endochronic models and most relevant references. The black square indicates that a thermodynamic formulation based on a suited pseudopotential was found for the associated group of models.

model with damage and is a direct extension of the results concerning the endochronic model without damage already discussed in Erlicher and Point (2006). Only plastically incompressible models are considered here, as they explain the main ideas, without introducing a too complex formalism. The extension to the general case is possible, but it is omitted for simplicity. The proposed analysis has an intrinsic interest, as it allows an easier comparison between endochronic models with damage and classical plasticity models with damage. Nonetheless, in the authors' opinion, another important reason justifies the interest toward this class of models: they represent the suitable theoretical basis for the analysis of the thermodynamic admissibility of the Bouc–Wen models with strength and stiffness degradation (see Bouc 1971; Wen 1976; Baber and Wen 1981; Casciati 1989; Karray and Bouc 1989). This was one of the main motivation at the origin of the present study, and the related developments about degrading Bouc–Wen models are presented in a companion paper (Erlicher and Bursi 2008).

After the introduction, the endochronic theory is presented in the second section: in the first part, standard endochronic models are described, whereas the second part concerns the definition of the flow rules of the extended endochronic theory, characterized by an additional scalar variable endowed with damage. The thermodynamic framework, with the definition of the suited pseudopotentials, is discussed in the following section and is supplemented by numerical examples. Then, a brief discussion about stability and uniqueness is made and the concluding remarks are given, where the topics dealt with in the companion paper (Erlicher and Bursi 2008) are pointed out.

Endochronic Models

Flow Rules of Plastically Incompressible ND-EC Models

The endochronic theory was first formulated by Valanis (1971), who suggested the use of a positive scalar variable ϑ, called the *intrinsic time scale*, in the definition of constitutive plasticity models. The evolution laws are described by convolution integrals involving past values of the strain $\boldsymbol{\varepsilon}$ and a suitable scalar function μ depending on ϑ, called *memory kernel*. When the memory kernel is exponential, the integral expressions can be rewritten as simple differential equations, the flow rules; in the case of an isotropic endochronic model without hardening or softening, called here ND-EC model (see Fig. 1), fulfilling the plastic incompressibility assumption, they read:

$$\begin{gathered}\mathrm{tr}(\boldsymbol{\sigma}) = 3K\,\mathrm{tr}(\boldsymbol{\varepsilon}), \quad \mathrm{dev}(\boldsymbol{\sigma}) = \mathbf{z} \\ \dot{\mathbf{z}} = 2G\,\mathrm{dev}(\dot{\boldsymbol{\varepsilon}}) - \beta \mathbf{z}\dot{\vartheta}\end{gathered} \tag{1}$$

where $\beta > 0$ (notice that β different from zero is needed to have a nonelastic behavior); the superposed dot indicates the time derivative; $\boldsymbol{\varepsilon}$=small strain tensor; $\boldsymbol{\sigma}$=Cauchy stress tensor; tr and dev=trace and deviatoric operators, respectively; K=bulk modulus; and G=shear modulus. The simplest choice for the *intrinsic time scale* flow indicated in Eq. (1) is $\dot{\vartheta} = \|\mathrm{dev}(\dot{\boldsymbol{\varepsilon}})\|$. It is interesting to note that relationships (1) are equivalent to

$$\begin{gathered}\boldsymbol{\sigma} = \mathbf{C}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \\ \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}^p) = 0, \quad \dot{\boldsymbol{\varepsilon}}^p = \beta \frac{\mathrm{dev}(\boldsymbol{\sigma})}{2G}\dot{\vartheta}\end{gathered} \tag{2}$$

where the trace of the plastic strain flow $\dot{\boldsymbol{\varepsilon}}^p$ is zero, consistent with the assumption of plastic incompressibility. $\mathbf{C} = (K - 2G/3)\mathbf{1} \otimes \mathbf{1} + 2G\mathbf{I}$=elasticity fourth-order tensor for isotropic materials; $\mathbf{1}$=second-order identity tensor; $\mathbf{I}$=fourth-order identity tensor and $\otimes$ represents the tensor product.

Flow Rules of Plastically Incompressible D-EC and DD-EC Models

An endochronic model with isotropic hardening or softening with plastically incompressible flow is defined as follows:

$$\begin{gathered}\mathrm{tr}(\boldsymbol{\sigma}) = 3K\,\mathrm{tr}(\boldsymbol{\varepsilon}), \quad \mathrm{dev}(\boldsymbol{\sigma}) = \mathbf{z} \\ \dot{\mathbf{z}} = 2G\,\mathrm{dev}(\dot{\boldsymbol{\varepsilon}}) - \beta \mathbf{z}\dot{\vartheta} \quad \text{with } \dot{\vartheta} = \frac{\dot{\zeta}}{g}\end{gathered} \tag{3}$$

where $g > 0$ is called the *hardening-softening function* (Bažant 1978). As stated by its name, the function g introduces isotropic hardening (or softening), which distinguishes this model (D-EC) from the basic ones presented in the previous section and indicated as ND-EC (see Fig. 1). In the classical endochronic formulations, g is a function of ζ, where ζ is the *intrinsic time measure*. A standard choice is $\dot{\zeta} = \|\mathrm{dev}(\dot{\boldsymbol{\varepsilon}})\|$ according to Valanis (1971). Another more general definition, leading to a cyclic behavior similar to that of the Prandtl–Reuss model when the positive parameter n is large enough, reads

$$\dot{\zeta} = \left(1 + \frac{\gamma}{\beta}\,\mathrm{sgn}(\mathbf{z}:\mathrm{dev}(\dot{\boldsymbol{\varepsilon}}))\right)|\mathbf{z}:\mathrm{dev}(\dot{\boldsymbol{\varepsilon}})|\|\mathbf{z}\|^{n-2} \tag{4}$$

with $\gamma \in [-\beta, \beta]$ in order to ensure the nonnegativity of $\dot{\zeta}$ and sgn=signum function. An important difference between Eq. (4) and the standard definition $\dot{\zeta} = \|\mathrm{dev}(\dot{\boldsymbol{\varepsilon}})\|$ is related to the product $\mathbf{z}$: $\mathrm{dev}(\dot{\boldsymbol{\varepsilon}})$, entailing $\dot{\zeta} = 0$ when the deviatoric strain increment is orthogonal to the stress. Like in the standard case, $\dot{\zeta}$ can be different from zero during unloading, i.e., when $\mathbf{z}$: $\mathrm{dev}(\dot{\boldsymbol{\varepsilon}}) < 0$. Eq. (4) shows that γ affects the difference between the loading and unloading values of the intrinsic time increment at a given stress $\mathbf{z}$. In particular, when $\gamma = \beta$ these increments are zero during unloading, while γ close to (and greater than) $-\beta$ leads to relatively small increments during loading, whereas $\dot{\zeta}$ is relatively large during unloading. The influence of n on the endochronic model behavior is discussed in the last section, with reference to the strain accumulation and the stress relaxation effects. According to Eqs. (3) and (4) and assuming $\beta+\gamma>0$, the norm of the tensor $\mathbf{z}(t)$ is bounded as follows:

$$\|\mathbf{z}(t)\| = \|\mathrm{dev}(\boldsymbol{\sigma}(t))\| < \sigma_u = \left(\frac{2G}{\beta+\gamma}\right)^{1/n} \tag{5}$$

for $t>0$, provided that $\|\mathbf{z}(0)\|<\sigma_u$. This inequality proves that a limit strength value exists and only concerns the deviatoric part of the stress $\boldsymbol{\sigma}$, consistently with the plastic incompressibility requirement. Eq. (5) also shows that this bounding stress depends on the parameters β, γ, and n.

Eq. (3) is equivalent to

$$\begin{gathered}\boldsymbol{\sigma} = \mathbf{C}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)\\ \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}^p) = 0, \quad \dot{\boldsymbol{\varepsilon}}^p = \frac{\mathrm{dev}(\boldsymbol{\sigma})}{2G/\beta}\frac{\dot{\zeta}}{g}\end{gathered} \tag{6}$$

From the last relationship in Eq. (6), it appears that the parameters β, γ, and n introduced in Eq. (4) affect the amplitude of the plastic strain flow, whereas the direction is always that of $\mathrm{dev}(\boldsymbol{\sigma})$.

A larger class of endochronic models can be defined by the following relationships:

$$\begin{gathered}\boldsymbol{\sigma} = (1-D)\mathbf{C}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)\\ \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}^p) = 0, \quad \dot{\boldsymbol{\varepsilon}}^p = \frac{1}{1-D}\frac{\mathrm{dev}(\boldsymbol{\sigma})}{2G/\beta}\frac{\dot{\zeta}}{g}\end{gathered} \tag{7}$$

where D=scalar variable introducing isotropic damage. The plasticity model with damage defined by Eq. (7) is named here the *extended endochronic model* and it belongs to the class of DD-EC models, as indicated in Fig. 1. Note that the stress is defined by introducing the factor $(1-D)$, consistently with the definition of *effective stress* and the *principle of strain equivalence* (Lemaitre and Chaboche 1990). Moreover, it can be observed that relationships (7) are equivalent to

$$\begin{gathered}\mathrm{tr}(\boldsymbol{\sigma}) = (1-D)3K\,\mathrm{tr}(\boldsymbol{\varepsilon}), \quad \mathrm{dev}(\boldsymbol{\sigma}) = \mathbf{z}\\ \dot{\mathbf{z}} = (1-D)2G\,\mathrm{dev}(\dot{\boldsymbol{\varepsilon}}) - \beta\mathbf{z}\dot{\vartheta} - \dot{D}\frac{\mathbf{z}}{1-D} \quad \text{with } \dot{\vartheta} = \frac{\dot{\zeta}}{g}\end{gathered} \tag{8}$$

which can be compared with Eq. (3).

A possible choice for $\dot{\zeta}$ is given by

$$\dot{\zeta} = \left(1 + \frac{\gamma}{\beta}\,\mathrm{sgn}(\mathbf{z}:\mathrm{dev}(\dot{\boldsymbol{\varepsilon}}))\right)|\mathbf{z}:\mathrm{dev}(\dot{\boldsymbol{\varepsilon}})|\|\mathbf{z}\|^{n-2}(1-D)^{1-n} \tag{9}$$

which represents a direct generalization of Eq. (4): the last factor depending on D and n is introduced in order to have an intrinsic time depending on the effective stress instead of the actual one, consistently with the strain equivalence postulate. An elastic with damage model can be defined by assuming $\dot{\zeta}=0$. In the authors' knowledge, the notions of pseudopotential and loading surface were never applied to the extended endochronic theory; therefore, these aspects are analyzed in detail in the next section.

Thermodynamic Framework for the Extended Endochronic Theory

The aim of this section is to define the Helmholtz free energy and the pseudopotential leading to the constitutive rules (7) or, equivalently, to Eq. (8). Under the assumption of isothermal and small transformations, the Helmholtz free energy density is chosen as follows:

$$\Psi = \Psi(\mathbf{v}) = \psi(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\zeta,D) + \mathbb{I}_{\mathbb{H}}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\zeta,D) \tag{10}$$

where $\mathbf{v}=(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\zeta,D)$=vector of state variables; $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$, and D were previously defined; ζ=scalar internal variable associated with isotropic hardening. For all the state variables, an initial zero value is assumed. The choice of ζ to indicate an internal variable might seem misleading, since the symbol ζ was also used in Eqs. (3)–(4) and (6)–(8) to define the *intrinsic time measure*. However, as it will be seen hereafter, this choice is the proper one, since for endochronic models ζ has simultaneously both meanings; ψ is the regular part of the Helmholtz energy; $\mathbb{I}_{\mathbb{H}}$ is the indicator function of the closed set $\mathbb{H}$: by definition, an indicator function is equal to 0 inside $\mathbb{H}$ and equal to $+\infty$ outside (Rockafellar 1969); the set $\mathbb{H}$ indicates the admissibility domain for the state variables $\mathbf{v}$ and should be introduced every time some conditions on state variables are to be imposed: for instance, it is equal to the interval $D\in[0,1]$ in order to impose the admissible values for the damage variable (Frémond 2002).

Once Ψ is known, the *nondissipative* thermodynamic forces $\mathbf{q}^{\mathrm{nd}}=(\boldsymbol{\sigma}^{\mathrm{nd}},\boldsymbol{\tau}^{\mathrm{nd}},R^{\mathrm{nd}},Y^{\mathrm{nd}})$ are defined as the gradient of $\psi(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\zeta,D)$

$$\boldsymbol{\sigma}^{\mathrm{nd}} := \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}, \quad \boldsymbol{\tau}^{\mathrm{nd}} := \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^p}, \quad R^{\mathrm{nd}} := \frac{\partial\psi}{\partial\zeta}, \quad Y^{\mathrm{nd}} := \frac{\partial\psi}{\partial D} \tag{11}$$

whereas the *nondissipative reaction forces* $\mathbf{q}^{\mathrm{ndr}}=(\boldsymbol{\sigma}^{\mathrm{ndr}},\boldsymbol{\tau}^{\mathrm{ndr}},R^{\mathrm{ndr}},Y^{\mathrm{ndr}})$ are given by

$$(\boldsymbol{\sigma}^{\mathrm{ndr}},\boldsymbol{\tau}^{\mathrm{ndr}},R^{\mathrm{ndr}},Y^{\mathrm{ndr}}) \in \partial\mathbb{I}_{\mathbb{H}}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p,\zeta,D) \tag{12}$$

where ∂=*subdifferential* operator (Rockafellar 1969). If the constraints imposed by $\mathbb{H}$ are fulfilled, the indicator function $\mathbb{I}_{\mathbb{H}}$ is zero and $\Psi(t)=\psi(t)$. This entails the identity of the time derivatives, viz. $\dot{\Psi}(t)=\dot{\psi}(t)+\mathbf{q}^{\mathrm{ndr}}\cdot\dot{\mathbf{v}}=\dot{\psi}(t)$. In other words, one has $\mathbf{q}^{\mathrm{ndr}}\cdot\dot{\mathbf{v}}=0$ for every instant t (Frémond 2002).

Due to the assumptions of isothermal and small transformations, the expression of the second principle reads

$$\Phi_1(t) = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \dot{\psi} \geqslant 0 \tag{13}$$

Eq. (13) states that the *intrinsic* (or *mechanical*) *dissipation* Φ_1 has to be nonnegative. Introducing the *dissipative* thermodynamic forces $\mathbf{q}^d=(\boldsymbol{\sigma}^d,\boldsymbol{\tau}^d,R^d,Y^d)$ as

$$\begin{gathered}\boldsymbol{\sigma}^d := \boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{nd}} - \boldsymbol{\sigma}^{\mathrm{ndr}}, \quad \boldsymbol{\tau}^d := -\boldsymbol{\tau}^{\mathrm{nd}} - \boldsymbol{\tau}^{\mathrm{ndr}}\\ R^d := -R^{\mathrm{nd}} - R^{\mathrm{ndr}}, \quad Y^d := -Y^{\mathrm{nd}} - Y^{\mathrm{ndr}}\end{gathered} \tag{14}$$

and substituting Eq. (14) into Eq. (13), one obtains:

$$\Phi_1(t) = \boldsymbol{\sigma}^d:\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\tau}^d:\dot{\boldsymbol{\varepsilon}}^p + R^d\dot{\zeta} + Y^d\dot{D} \geqslant 0 \tag{15}$$

In order to fulfill the inequality (15), the flows of the state variables $\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\varepsilon}}^p,\dot{\zeta}$, and $\dot{D}$ have to be suitably correlated with the *dissipative* thermodynamic forces $\boldsymbol{\sigma}^d,\boldsymbol{\tau}^d,R^d$, and Y^d. Therefore, some additional *complementarity rules* need to be defined: usually, a scalar nonnegative function called *pseudopotential*

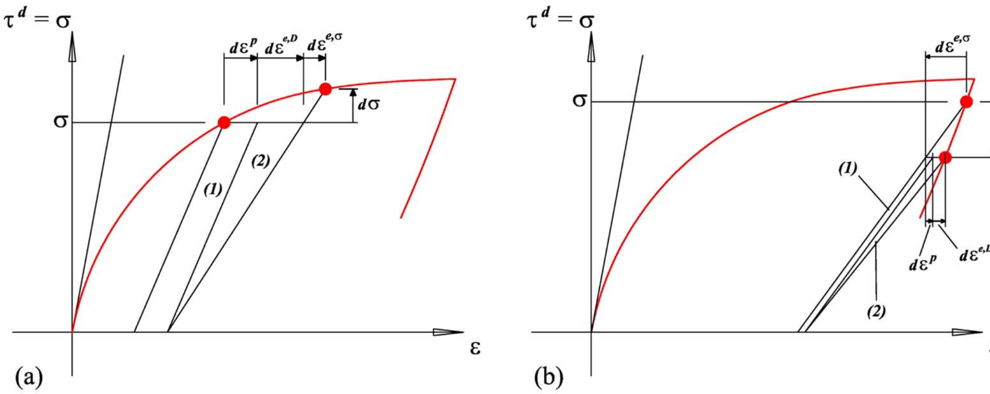

Fig. 2. Increment de_p of the energy dissipated by plastic strain (1) and the increment de_D of the energy dissipated by damage (2). The increments of the elastic and plastic strain as well as of the stress are also schematically illustrated for (a) the loading phase and (b) the unloading phase.

$$\phi = \phi(\dot{\mathbf{v}}';\mathbf{v};\boldsymbol{\rho}) \quad \text{such that } \phi(\mathbf{0};\mathbf{v};\boldsymbol{\rho}) = 0 \quad \text{for all } \mathbf{v} \text{ and } \boldsymbol{\rho} \tag{16}$$

is introduced and the dissipative forces $\mathbf{q}^d = (\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d, R^d, Y^d)$ are derived imposing the so-called *generalized normality assumption* on it. Equivalently, one can define the flow rules $\dot{\mathbf{v}}$ by imposing the generalized normality assumption on the *dual* pseudopotential ϕ^*, which is the Legendre–Fenchel transform of ϕ (Rockafellar 1969). This last method will be explicitly exploited herein. The generic flow $\dot{\mathbf{v}}'$ is noted with "prime," while the actual flow at the present state is noted with $\dot{\mathbf{v}}$. As a matter of fact, the pseudopotential is assumed to depend on the present value of state variables $\mathbf{v}$ and on some additional parameters collected in the vector $\boldsymbol{\rho}$. These parameters may be any quantity related to the past history of the material (Frémond 2002). For instance, one may have $\boldsymbol{\rho}(\mathbf{x}) = (e(\mathbf{x}), \|\boldsymbol{\varepsilon}(\mathbf{x})\|_{\max}) = (e(\mathbf{x}), \max_{0\leqslant t'\leqslant t}\|\boldsymbol{\varepsilon}(\mathbf{x},t')\|)$, where e = dissipated energy per unit volume at the point $\mathbf{x}$ of the body volume and $\|\boldsymbol{\varepsilon}(\mathbf{x})\|_{\max}$ = maximum (from $t'=0$ to the present state $t'=t$) of the strain norm at the same point.

When no viscous effect occurs, the case of plasticity with damage is recovered. This corresponds to choose a pseudopotential ϕ independent from $\dot{\boldsymbol{\varepsilon}}$, entailing $\boldsymbol{\sigma}^d = \mathbf{0}$; for a detailed derivation of these relationships, see, for instance, Erlicher and Point (2006). Moreover, "plastic flow may occur without damage and damage may occur without appreciable macroscopic plastic flow" (Lemaitre and Chaboche 1990). Therefore, the inequality (15) with $\boldsymbol{\sigma}^d = \mathbf{0}$ "must be split in two independent inequalities":

$$\dot{e}_p \coloneqq \boldsymbol{\tau}^d : \dot{\boldsymbol{\varepsilon}}^p + R^d \dot{\zeta} \geqslant 0, \quad \dot{e}_D \coloneqq Y^d \dot{D} \geqslant 0 \tag{17}$$

The two scalar quantities $\dot{e}_p$ and $\dot{e}_D$, respectively, define the rate of energy per unit volume dissipated by plasticity-related phenomena and by damage phenomena; see Fig. 2. Their sum $\dot{e} = \dot{e}_p + \dot{e}_D$ is the rate of the total dissipated energy per unit volume and coincides with the intrinsic dissipation Φ_1. The restrictions imposed by these two inequalities are more severe than the original unique inequality of Clausius–Duhem (15). However, they are usually adopted as basic thermodynamic criterion for the formulation of plasticity models with damage (Lemaitre and Chaboche 1990). This assumption will be adopted hereafter. Taking into account the inequalities (17), the pseudopotential is supposed to split into two pseudopotentials ϕ_D and ϕ_p, respectively related to damage and plastic flow:

$$\phi(\dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}', \dot{D}'; \mathbf{v}; \boldsymbol{\rho}) = \phi_D(\dot{D}'; \mathbf{v}; \boldsymbol{\rho}) + \phi_p(\dot{\boldsymbol{\varepsilon}}^{p\prime}, \dot{\zeta}'; \mathbf{v}; \boldsymbol{\rho}) \tag{18}$$

In the following sections, the Helmholtz free energy, the pseudopotentials ϕ_D and ϕ_p, as well as their Legendre–Fenchel transforms, are formulated for the endochronic model with damage (7).

Helmholtz Free Energy

According to Eq. (10), for the DD-EC models one has the following Helmholtz free energy:

$$\Psi(\mathbf{v}) = \psi(\mathbf{v}) + \mathbb{I}_{\mathbb{H}}(\mathbf{v}) = (1-D)\tfrac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p):\mathbf{C}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \mathbb{I}_{\mathbb{H}}(\mathbf{v}) \tag{19}$$

In this paper, two cases are considered

$$\mathbb{H} = \left\{ \begin{aligned} &(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, D) \quad \text{such that } D \geqslant 0, D \leqslant 1 \\ &\text{and } (1-D)^s R(\tilde{\mathbf{v}}, \boldsymbol{\rho}) - r_0 \leqslant 0 \end{aligned} \right\} \tag{20}$$

where s and r_0 = positive parameters; $\tilde{\mathbf{v}} = (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta)$ collects all state variables except D; and $R = R(\tilde{\mathbf{v}}, \boldsymbol{\rho})$ is a nonnegative function called *source of damage*. The first two conditions on D impose the minimum and the maximum values for this variable. As it will be seen, the third condition in Eq. (20) is strictly related to the definition of the damage limit surface. The second case is characterized by a different assumption

$$\mathbb{H} = \{(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \zeta, D) \quad \text{such that } D \geqslant 0, D \leqslant 1\} \tag{21}$$

where only the two basic inequalities on D are retained.

Making use of Eqs. (14), (19), and (20), and of the pseudopotential (25), i.e., Definition 1 of ϕ_D given in the following section, it is possible to prove that the assumption $\mathbf{q}^{\mathrm{ndr}} = \mathbf{0}$ is admissible. The same holds for the model defined by Eqs. (19), (21), and (34) (Definition 2 of ϕ_D). For brevity, the details of this proof, are omitted. As a result, the thermodynamic forces fulfill the following relationships:

$$\boldsymbol{\sigma}^{\mathrm{nd}} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} = (1-D)\mathbf{C}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = \boldsymbol{\sigma} - \boldsymbol{\sigma}^d = \boldsymbol{\sigma} \tag{22a}$$

$$\boldsymbol{\tau}^{\mathrm{nd}} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^p} = -(1-D)\mathbf{C}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = -\boldsymbol{\tau}^d \tag{22b}$$

$$R^{\mathrm{nd}}=\frac{\partial\psi}{\partial\zeta}=0=-R^{d} \tag{22c}$$

$$Y^{\mathrm{nd}}=\frac{\partial\psi}{\partial D}=-\frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{p}):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{p})=-Y^{d} \tag{22d}$$

Moreover using Eq. (22) and supposing $D<1$, the energy dissipation rate reads

$$\dot{e}=\dot{e}_{p}+\dot{e}_{D}=\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{p}+\frac{1}{2}\boldsymbol{\sigma}:\frac{\mathbf{C}^{-1}:\boldsymbol{\sigma}}{(1-D)^{2}}\dot{D}=\mathrm{dev}(\boldsymbol{\sigma}):\mathrm{dev}(\dot{\boldsymbol{\varepsilon}}^{p})+\frac{\mathrm{tr}(\boldsymbol{\sigma})}{3}\mathrm{tr}(\dot{\boldsymbol{\varepsilon}}^{p})+\frac{1}{2}\left(\frac{\mathrm{dev}(\boldsymbol{\sigma}):\mathrm{dev}(\boldsymbol{\sigma})}{2G}+\frac{(\mathrm{tr}(\boldsymbol{\sigma}))^{2}}{9K}\right)\frac{\dot{D}}{(1-D)^{2}} \tag{23}$$

Taking the time-derivative of (22*a*) and recalling that no viscous effect is considered ($\boldsymbol{\sigma}^{d}=\mathbf{0}$), one obtains

$$\dot{\boldsymbol{\varepsilon}}=\dot{\boldsymbol{\varepsilon}}^{p}+\dot{\boldsymbol{\varepsilon}}^{e}=\dot{\boldsymbol{\varepsilon}}^{p}+\mathbf{C}^{-1}:\frac{\dot{\boldsymbol{\sigma}}}{1-D}+\mathbf{C}^{-1}:\frac{\boldsymbol{\sigma}\dot{D}}{(1-D)^{2}}:=\dot{\boldsymbol{\varepsilon}}^{p}+\dot{\boldsymbol{\varepsilon}}^{e,\sigma}+\dot{\boldsymbol{\varepsilon}}^{e,D} \tag{24}$$

where $\dot{\boldsymbol{\varepsilon}}^{e,\sigma}$=elastic strain rate at constant damage and $\dot{\boldsymbol{\varepsilon}}^{e,D}$ =elastic strain rate at constant stress. It follows that $\dot{e}_{D}=\frac{1}{2}(\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{e,D})$. Note that in general the endochronic theory may present nonzero energy rates $\dot{e}_{p}$ and $\dot{e}_{D}$ also during unloading phases; see Fig. 2(b) in this respect.

Pseudopotential for the Damage Flow

The formalism of the loading function f_D, as well as the pseudopotential $\phi_D^*=\mathbb{I}_{f_D\leqslant 0}$, can be used to express the damage evolution (Lemaitre and Chaboche 1990; Salari et al. 2004; Nedjar 2001; Frémond 2002). We present herein a well-known damage evolution rule by using both pseudopotentials ϕ_D and its dual ϕ_D^*. Then, a discussion is done about a novel pseudopotential leading to a damage evolution where $\dot{D}$ may be different from 0 also during unloading phases. In detail, the main difference between the two cases is related to the role of the damage limit surface. Standard damage evolution rules, viz. Definition 1, are characterized by the possibility for the actual state point to be inside the damage domain delimited by this limit surface; in this situation and in particular during unloading phases, damage increments are null. Conversely, in the damage evolution which we propose here, i.e., Definition 2, the present state point is forced to be always on the damage limit surface also during unloading phases.

Definition 1 of ϕ_D

Let us begin with the following pseudopotential, associated with the Helmholtz free energy (19)–(20):

$$\phi_{D}(\dot{D}';\mathbf{v};\boldsymbol{\rho})=\left[\tfrac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{p}):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{p})-(1-D)^{s}R(\tilde{\mathbf{v}},\boldsymbol{\rho})+r_{0}\right]\dot{D}'+\mathbb{I}_{\mathbb{D}_{D}}(\dot{D}') \tag{25}$$

and

$$\mathbb{D}_{D}=\{\dot{D}'\ \text{such that}\ \dot{D}'\geqslant 0\} \tag{26}$$

The pseudopotential ϕ_D is the sum of a *regular* part, proportional to $\dot{D}'$ and of the indicator function $\mathbb{I}_{\mathbb{D}_D}$. The term multiplying $\dot{D}'$ in the regular part of ϕ_D is always nonnegative, by virtue of the third condition defining $\mathbb{H}$ in Eq. (20). The regular part of ϕ_D, considered for the actual flow $\dot{D}'=\dot{D}$, represents the rate of dissipated energy $\dot{e}_D$. Eqs. (25) and (26) allow a large number of standard damage evolution rules to be represented, according to the specific definition of $R(\tilde{\mathbf{v}},\boldsymbol{\rho})$. An interesting example is

$$R=R(\boldsymbol{\varepsilon})=\tfrac{1}{2}(2G\boldsymbol{\varepsilon}^{e,+}:\boldsymbol{\varepsilon}^{e,+}+\lambda([\mathrm{tr}(\boldsymbol{\varepsilon}^{e})]^{+})^{2}) \tag{27}$$

where $\lambda=K-2G/3$=Lamé constant. For a scalar x, $[x]^{+}:=\langle x\rangle$, where the angular brackets are the McCauley brackets. The positive part $\boldsymbol{\varepsilon}^{e,+}$ of the tensor $\boldsymbol{\varepsilon}^{e}$ is obtained after diagonalization. Other definitions for R can be used; see, among others, Nedjar (2001) and Salari et al. (2004).

In order to derive the damage flow, it is convenient to consider the Legendre–Fenchel transform of ϕ_D, which reads

$$\phi_{D}^{*}(Y^{d'};\mathbf{v};\boldsymbol{\rho})=\sup_{\dot{D}'\in\mathbb{D}_{D}}\left[\left(Y^{d'}-\tfrac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{p}):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{p})+(1-D)^{s}R(\tilde{\mathbf{v}},\boldsymbol{\rho})-r_{0}\right)\dot{D}'\right]=\mathbb{I}_{\mathbb{E}_{D}}(Y^{d'};\mathbf{v};\boldsymbol{\rho}) \tag{28}$$

where $\mathbb{E}_{D}=\{Y^{d'}$ such that $f_{D}(Y^{d'};\mathbf{v};\boldsymbol{\rho})\leqslant 0\}$=damage loading domain and the corresponding loading function is

$$f_{D}(Y^{d'};\mathbf{v};\boldsymbol{\rho})=Y^{d'}-\tfrac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{p}):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{p})+(1-D)^{s}R(\tilde{\mathbf{v}},\boldsymbol{\rho})-r_{0}:=Y^{d'}-Y_{\max}^{d'} \tag{29}$$

By using the normality assumption, the relevant damage flow rule reads

$$\dot{D}=\dot{\lambda}_{D}\frac{\partial f_{D}(Y^{d})}{\partial Y^{d'}}=\dot{\lambda}_{D} \tag{30}$$

with $f_{D}(Y^{d};\mathbf{v};\boldsymbol{\rho})\dot{\lambda}_{D}=0$, $\dot{\lambda}_{D}\geqslant 0$, $f_{D}(Y^{d};\mathbf{v};\boldsymbol{\rho})\leqslant 0$.

At the actual state, it holds $Y^{d'}=Y^{d}=-Y^{\mathrm{nd}}$, and therefore

$$f_{D}=f_{D}(Y^{d};\mathbf{v};\boldsymbol{\rho})=(1-D)^{s}R(\tilde{\mathbf{v}},\boldsymbol{\rho})-r_{0}\leqslant 0 \tag{31}$$

which is the damage limit surface, but also is one of the conditions defining the set $\mathbb{H}$. It becomes evident that the positive constant r_0 is the initial damage threshold. The Kuhn–Tucker conditions state that $f_D<0$ implies no damage increment, whereas $f_{D}(Y^{d};\mathbf{v};\boldsymbol{\rho})=0$ corresponds to a damage increment which can be computed by enforcing the consistency condition

$$\dot{f}_{D}=(1-D)^{s}\left(\frac{\partial R(\tilde{\mathbf{v}},\boldsymbol{\rho})}{\partial\tilde{\mathbf{v}}}\dot{\tilde{\mathbf{v}}}+\frac{\partial R(\tilde{\mathbf{v}},\boldsymbol{\rho})}{\partial\boldsymbol{\rho}}\dot{\boldsymbol{\rho}}\right)-R(\tilde{\mathbf{v}},\boldsymbol{\rho})s(1-D)^{s-1}\dot{D}=0 \tag{32}$$

leading to the explicit expression of the damage flow

$$\dot{D}=H(f_{D})\left(\frac{\partial R(\tilde{\mathbf{v}},\boldsymbol{\rho})}{\partial\tilde{\mathbf{v}}}\dot{\tilde{\mathbf{v}}}+\frac{\partial R(\tilde{\mathbf{v}},\boldsymbol{\rho})}{\partial\boldsymbol{\rho}}\dot{\boldsymbol{\rho}}\right)\frac{1-D}{sR(\tilde{\mathbf{v}},\boldsymbol{\rho})} \tag{33}$$

where H=Heaviside function. The presence of the Heaviside function in the damage flow definition indicates that damage increments are zero during unloading phases. Note that Eq. (33) entails that the limit condition $D=1$ is never reached.

Fig. 3 illustrates some loading-unloading cycles of an elastic with damage model ($\dot{\zeta}=0$). The uniaxial stress is considered, viz. all the components of the Cauchy tensor are supposed to be null, except σ_{11}. The parameter values represent a hypothetical material for which the Young modulus E=35,000 MPa and the Poisson ratio ν=0.18 are close to those of concrete; damage is defined by Eqs. (20) and (27), with s=2.5 and r_0=1.2*e* −05 MJ/m^3; see also the numerical examples in Nedjar (2001).

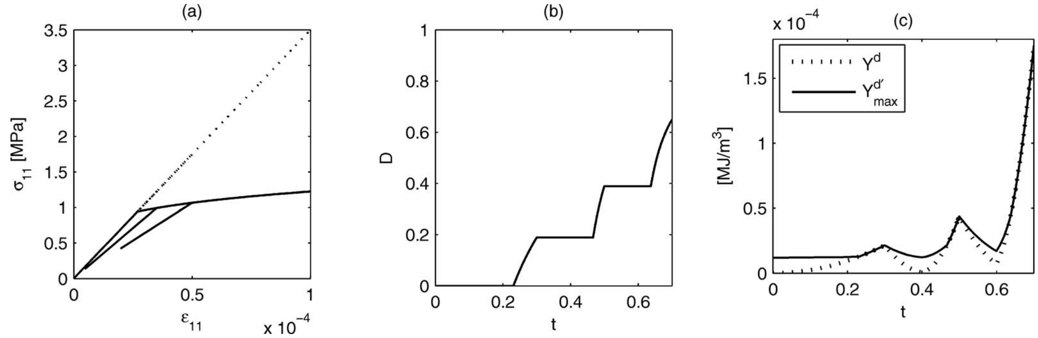

Fig. 3. Elastic with damage model. Uniaxial behavior. (a) The stress–strain evolution. The dotted line indicates the effective stress $\sigma_{11}/(1-D)$. (b) The damage evolution, with the damage increasing only during loading phases. (c) Time evolution of Y^d, i.e., the actual value of the thermodynamic force $Y^{d'}$, and of $Y^{d'}_{\max}$, i.e., the upper bound of $Y^{d'}$ defined in Eq. (29).

Together with the stress-strain and damage evolution of this model, Fig. 3 depicts the evolution of $Y^d = 1/2(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)$, i.e., the actual value of $Y^{d'}$, and of the quantity $Y^{d'}_{\max} = 1/2(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)+r_0-(1-D)^s R(\boldsymbol{\varepsilon})$, defining the upper limit of $Y^{d'}$ according to Eq. (29). When these two curves are superposed, the damage increases.

Definition 2 of ϕ_D

Unfortunately, a definition of $\dot{D}$ of type (33), deriving from the pseudopotential (25) and condition (20), is not able to represent the case of damage increasing during both loading and unloading phases, owing to the condition $f_D \leqslant 0$. We recall that the case of damage increasing during unloading may occur in Bouc–Wen models with stiffness degradation (Erlicher and Bursi 2008). A damage pseudopotential, simpler than Eq. (25), is more suited

$$\phi_D(\dot{D}';\mathbf{v}) = \left[\tfrac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)\right]\dot{D}' + \mathbb{I}_{\mathbb{D}_D}(\dot{D}') \qquad (34)$$

with $\mathbb{D}_D$ still provided by Eq. (26) and with the conditions on the damage state variable defined in Eq. (21). As already observed, it is possible to prove that the assumption $\mathbf{q}^{\text{ndr}}=\mathbf{0}$ is admissible also for this Definition 2 of the damage pseudopotential. The dual pseudopotential becomes $\phi^*_D(Y^{d'};\mathbf{v}) = \mathbb{I}_{\mathbb{E}_D}(Y^{d'};\mathbf{v})$ where $\mathbb{E}_D = \{Y^{d'}$ such that $f_D(Y^{d'};\mathbf{v}) \leqslant 0\}$ is the corresponding damage loading domain, with the damage loading function

$$f_D(Y^{d'};\mathbf{v}) = Y^{d'} - \tfrac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p) \coloneqq Y^{d'} - Y^{d'}_{\max} \qquad (35)$$

At the actual state, $Y^{d'} = Y^d = -Y^{\text{nd}}$ and therefore $f_D(Y^d;\mathbf{v})=0$ at every instant. Therefore, relationship (30) reduces to $\dot{D} = \dot{\lambda}_D \partial f_D/\partial Y^{d'} = \dot{\lambda}_D$, with $\dot{\lambda}_D \geqslant 0$. Moreover, $\dot{D} = \dot{\lambda}_D$ can no longer be computed by the consistency condition, fulfilled as an identity at every instant. Hence, it must be rather defined by an additional condition. Any definition ensuring rate-independence, consistent with Eq. (21) and fulfilling $\dot{D} = \dot{\lambda}_D \geqslant 0$ is admissible, even though is characterized by nonzero damage increments during unloading phases.

Pseudopotential for the Plastic Flow

The usual method to define associated plastic flows is based on the notion of *loading function*, indicated here by f_p, as well as on the normality assumption. Another equivalent formalism is based on the use of the dual pseudopotential $\phi^*_p = \mathbb{I}_{f_p \leqslant 0}$ (Moreau 1970). A third way to formulate plasticity models is based on the pseudopotential ϕ_p, Legendre–Fenchel conjugate of ϕ^*_p (Frémond 2002; Ziegler and Wehrli 1987; Houlsby and Puzrin 2000; Erlicher and Point 2006). The advantage of using the formalism based on f_p (or $\phi^*_p = \mathbb{I}_{f_p \leqslant 0}$) is essentially simplicity. Moreover, when a nonassociated flow is to be defined, the simple introduction of a second function g_p called plastic potential matches this purpose. Nonetheless, for some nonclassical plasticity theories, like endochronic theory and generalized plasticity (Lubliner et al. 1993), it is not straightforward to provide a proper definition of the loading function f_p. It was proven by Erlicher and Point (2006) that for these plasticity theories (without damage) a way to define the loading function is to start from the definition of the pseudopotential ϕ_p, to compute the dual potential ϕ^*_p and then to derive f_p. An important point is the additional dependence of ϕ_p, and therefore of ϕ^*_p and the loading function too, on the *state variables*. This dependence is only optional for standard plasticity theories but is essential both for the endochronic theory and the generalized plasticity. Moreover, we notice that some models with nonassociated flow also admit a representation based on the definition of a suited pseudopotential ϕ_p, depending on state variables. The example of a nonassociated Drucker–Prager model can be found in Erlicher and Point (2005); in particular, it is shown that a suited pseudopotential ϕ_p leads to a *modified* loading function which plays both roles of the traditional loading function and of the plastic potential.

For the endochronic models with damage, the plasticity pseudopotential is defined as follows:

$$\phi_p(\dot{\boldsymbol{\varepsilon}}^{p'},\dot{\zeta}';\mathbf{v};\boldsymbol{\rho}) = (1-D)\frac{\|\text{dev}(\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p))\|^2}{2G/\beta}\frac{\dot{\zeta}'}{g(\mathbf{v},\boldsymbol{\rho})} + \mathbb{I}_{\mathbb{D}}(\dot{\boldsymbol{\varepsilon}}^{p'},\dot{\zeta}';\mathbf{v};\boldsymbol{\rho}) \qquad (36)$$

where $\mathbb{I}_{\mathbb{D}}$ = indicator function of the convex set

$$\mathbb{D} = \left\{\begin{array}{l} (\dot{\boldsymbol{\varepsilon}}^{p'},\dot{\zeta}') \quad \text{such that} \\ \text{tr}(\dot{\boldsymbol{\varepsilon}}^{p'}) = 0, \quad \dot{\zeta}' \geqslant 0, \quad \text{and} \\ \dot{\boldsymbol{\varepsilon}}^{p'} = \dfrac{\text{dev}(\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p))}{2G/\beta}\dfrac{\dot{\zeta}'}{g(\mathbf{v},\boldsymbol{\rho})} \end{array}\right\} \qquad (37)$$

[see Fig. 4(a)]. The first equality in $\mathbb{D}$ imposes the plastic incompressibility of the flow. Moreover, as D is supposed to be less or equal to one and $g = g(\mathbf{v},\boldsymbol{\rho})$, the hardening-softening function, is positive by assumption, the second condition in $\mathbb{D}$ ensures the nonnegativity of ϕ_p. Therefore, the standard properties of ϕ_p, viz. nonnegativity, convexity, and positive homogeneity of order 1,

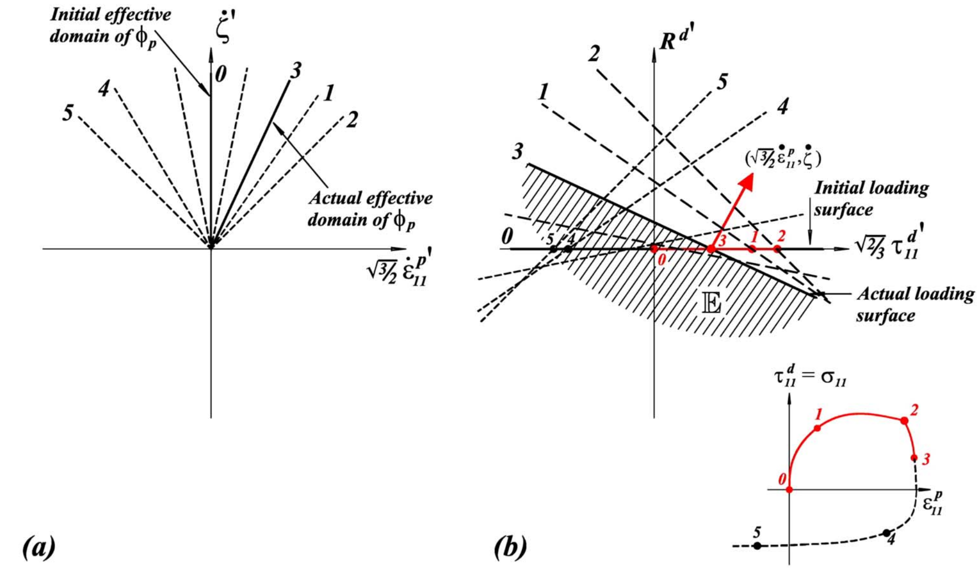

Fig. 4. Endochronic model with damage in the uniaxial stress regime. (a) Several configurations of the effective domain $\mathbb{D}$ of the pseudopotential ϕ_p. (b) Corresponding configurations of the loading domain $\mathbb{E}$, associated with the loading function f_p. The actual stress point always occurs at $R^{d'}=R^d=0$ and, at this point, the flows $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\zeta}$ are defined by the normality condition.

are matched. The third condition in $\mathbb{D}$ gives the plastic flow and is consistent with Eq. (7). It can be proven that when $\dot{\boldsymbol{\varepsilon}}^{p'}=\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\zeta}'=\dot{\zeta}$, i.e. when the actual flows are considered, the first term of the sum in Eq. (36) represents the rate of energy $\dot{e}_p$ dissipated by the plastic flow, defined in Eq. (17) for the general case. Note that the pseudopotential has an additional dependence on the state variables *and* on the past-history dependent parameters collected in $\boldsymbol{\rho}$.

The dual dissipation potential ϕ_p^* is obtained by the Legendre–Fenchel transformation of ϕ_p (Rockafellar 1969). As ϕ_p is positively homogeneous of order 1, then ϕ_p^* is an indicator function:

$$\begin{aligned}\phi_p^*(\boldsymbol{\tau}^{d'},R^{d'};\mathbf{v};\boldsymbol{\rho}) &= \sup_{(\dot{\boldsymbol{\varepsilon}}^{p'},\dot{\zeta}')\in\mathbb{D}} (\boldsymbol{\tau}^{d'}:\dot{\boldsymbol{\varepsilon}}^{p'}+R^{d'}\dot{\zeta}'-\phi_p(\dot{\mathbf{v}}';\mathbf{v};\boldsymbol{\rho}))\\ &= \sup_{(\dot{\boldsymbol{\varepsilon}}^{p'},\dot{\zeta}')\in\mathbb{D}}\Bigg(\mathrm{dev}(\boldsymbol{\tau}^{d'}):\frac{\mathrm{dev}(\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p))}{2Gg(\mathbf{v},\boldsymbol{\rho})/\beta}\dot{\zeta}'+R^{d'}\dot{\zeta}'\\ &\qquad -(1-D)\frac{\|\mathrm{dev}(\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p))\|^2}{2Gg(\mathbf{v},\boldsymbol{\rho})/\beta}\dot{\zeta}'\Bigg)\\ &= \mathbb{I}_{\mathbb{E}}(\boldsymbol{\tau}^{d'},R^{d'};\mathbf{v};\boldsymbol{\rho})\end{aligned} \tag{38}$$

The indicator function $\mathbb{I}_{\mathbb{E}}$ is associated with the convex set $\mathbb{E}=\{(\boldsymbol{\tau}^{d'},R^{d'})$ such that $f_p(\boldsymbol{\tau}^{d'},R^{d'};\mathbf{v};\boldsymbol{\rho})\leqslant 0\}$ [see Fig. 4(b)] with

$$\begin{aligned}f_p(\boldsymbol{\tau}^{d'},R^{d'};\mathbf{v};\boldsymbol{\rho}) &= \mathrm{dev}(\boldsymbol{\tau}^{d'}):\frac{\mathrm{dev}(\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p))}{2Gg(\mathbf{v},\boldsymbol{\rho})/\beta}\\ &\quad -(1-D)\frac{\|\mathrm{dev}(\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p))\|^2}{2Gg(\mathbf{v},\boldsymbol{\rho})/\beta}+R^{d'}\end{aligned} \tag{39}$$

The function f_p is the *loading function* for an endochronic model with plastic incompressibility and with isotropic damage. It is associated with the loading domain $\mathbb{E}$. If the past-history parameter $\boldsymbol{\rho}$ is a scalar equal to e_p, the plastic dissipated energy, then a *work-hardening* behavior is defined, in the sense that the loading function evolves with the plastic dissipated energy. A different approach to define work-hardening plasticity models was proposed by Ristinmaa (1999).

The generalized normality conditions imposed on ϕ_p^* leads to

$$\dot{\boldsymbol{\varepsilon}}^p=\dot{\lambda}\frac{\partial f_p(\boldsymbol{\tau}^d,R^d;\mathbf{v};\boldsymbol{\rho})}{\partial\boldsymbol{\tau}^{d'}}=\dot{\lambda}\frac{\mathrm{dev}(\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p))}{2Gg(\mathbf{v},\boldsymbol{\rho})/\beta}$$

$$\dot{\zeta}=\dot{\lambda}\frac{\partial f_p(\boldsymbol{\tau}^d,R^d;\mathbf{v};\boldsymbol{\rho})}{\partial R^{d'}}=\dot{\lambda} \tag{40}$$

$$\dot{\lambda}f_p(\boldsymbol{\tau}^d,R^d;\mathbf{v};\boldsymbol{\rho})=0 \quad f_p(\boldsymbol{\tau}^d,R^d;\mathbf{v};\boldsymbol{\rho})\leqslant 0, \quad \dot{\lambda}\geqslant 0$$

where the last three inequalities are the Kuhn–Tucker conditions. The plastic flow defined in Eq. (7) is retrieved. Note that the derivatives of f_p defining $\dot{\boldsymbol{\varepsilon}}^{p'}$ and $\dot{\zeta}$ are taken with respect to the generic variables $\boldsymbol{\tau}^{d'}$ and $R^{d'}$, but they are computed at the present state $\boldsymbol{\tau}^{d'}=\boldsymbol{\tau}^d$ and $R^{d'}=R^d$. In summary, the usual notions of plastic multiplier and loading surface have been defined for an endochronic model with damage. This kind of thermodynamic formulation for endochronic models is quite innovative and has been first presented by Erlicher and Point (2006), for the case of no damage. As was pointed out in that paper, an important property characterizing endochronic models is the fact that at the actual state, the loading function f_p is always zero: for this reason, the consistency condition $\dot{f}_p=0$ is always fulfilled as an identity and cannot be used to compute the plastic multiplier $\dot{\lambda}$. This is also true in this case, where the actual state is $(\boldsymbol{\tau}^d,R^d)=((1-D)\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p),0)$. As a result, the Kuhn–Tucker conditions reduce to $\dot{\lambda}=\dot{\zeta}\geqslant 0$, where $\dot{\zeta}$ is the flow of the internal variable associated with R^d and, using the language of the endochronic theory, is also the flow of the intrinsic time measure; it can be freely defined, provided that it is nonnegative and that rate-independence is guaranteed. As already observed, the standard choice is $\dot{\zeta}=\|\mathrm{dev}(\dot{\boldsymbol{\varepsilon}})\|$.

Fig. 5 illustrates an example of uniaxial behavior of an endochronic plasticity model with damage. The parameters of the elastic phase and of damage (Definition 1) are the same as those of Fig. 3. In addition, $g=1$, $\dot{\zeta}$ is given by Eq. (9) with

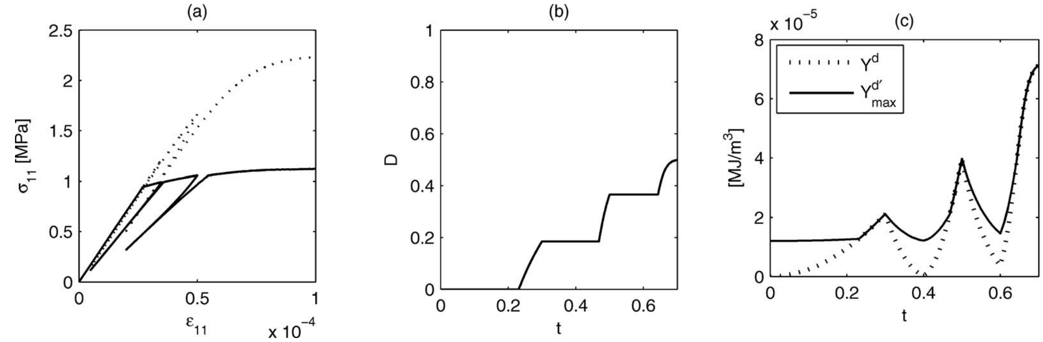

Fig. 5. Uniaxial behavior of an endochronic plasticity model with damage (Definition 1): (a) the stress–strain evolution. The dotted line indicates the effective stress $\sigma_{11}/(1-D)$; (b) damage evolution; and (c) see Fig. 3(c).

$n=5$, $\beta=2{,}834.9\ \mathrm{MPa}^{1-n}$ and $\gamma/\beta=-0.5$; as a result, $\sigma_u=(2G/(\beta+\gamma))^{1/n}=2.25\sqrt{2/3}=1.8371$ MPa, where σ_u=upper limit of $\|\mathrm{dev}(\boldsymbol{\tau}^d/(1-D))\|=\|\mathrm{dev}(\boldsymbol{\sigma}/(1-D))\|=\sqrt{2/3}\sigma_{11}/(1-D)$ when $g=1$.

In the example of Fig. 6, the damage is defined by the rule $D=1-1/(1+c_\eta e_p)$, with $c_\eta=1{,}500\ \mathrm{m}^3\,\mathrm{MJ}^{-1}$ (Definition 2). The parameter c_η indicates the sensitivity of damage to the energy e_p dissipated by plasticity. If c_η is large, the damage increment at a given e_p-value is larger than in the case of small c_η. Young's modulus and the Poisson ratio are the same as in the previous figures. The parameters defining the intrinsic time flow (9) are: $n=15$, $\beta=16.1846\ \mathrm{MPa}^{1-n}$, and $\gamma/\beta=-0.8$; as a result, $\sigma_u=(2G/(\beta+\gamma))^{1/n}=2.25\sqrt{2/3}=1.8371$ MPa. Moreover, the hardening function is defined as $g=(1+\|\mathrm{dev}(\boldsymbol{\varepsilon}(t'))\|_{\max_{t'\in[0,t]}}/\varepsilon_u)^n$, where t=present time and $\varepsilon_u=0.0002$. Fig. 6(d) depicts the evolution of $Y^d=\frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)$, i.e., the actual value of $Y^{d'}$. According to Eq. (35), this quantity is also equal to $Y^{d'}_{\max}$, which is the upper limit of $Y^{d'}$. The typical endochronic behavior with plastic strains increasing during unloading phases is highlighted in Fig. 6(b). As a result, owing to the damage rule depending on the dissipated plastic energy, also the damage slightly increases during the unloading phases: observe the damage evolution after $t=0.3$ and 0.5, which are the instants where unloading phases begin.

Brief Discussion about Stability and Uniqueness

It is well known that standard endochronic models violate the Drucker's postulate and the Ilyushin's postulate, (see e.g., Sandler 1978). As a result, inelastic strains may continuously increase if a cyclic stress of constant and arbitrarily small amplitude is imposed around a given static stress [Fig. 7(a)]. Dually, a stress relaxation occurs when cycling straining is imposed [Fig. 8(a)]. The parameters used for the numerical simulations of Figs. 7 and 8 are: $E=35{,}000$ MPa, $\nu=0.18$, $g=1$, $\gamma/\beta=-0.5$, whereas β has a value such that $(2G/(\beta+\gamma))^{1/n}=1.8371$ MPa, for the given n values used in the figures. The strain accumulation entails a violation of a Lyapunov-type stability condition. For this reason, endochronic theory have been repeatedly criticized in the past years. However, Bažant (1978, p. 705) showed that endochronic models do fulfill some weaker physically motivated stability conditions.

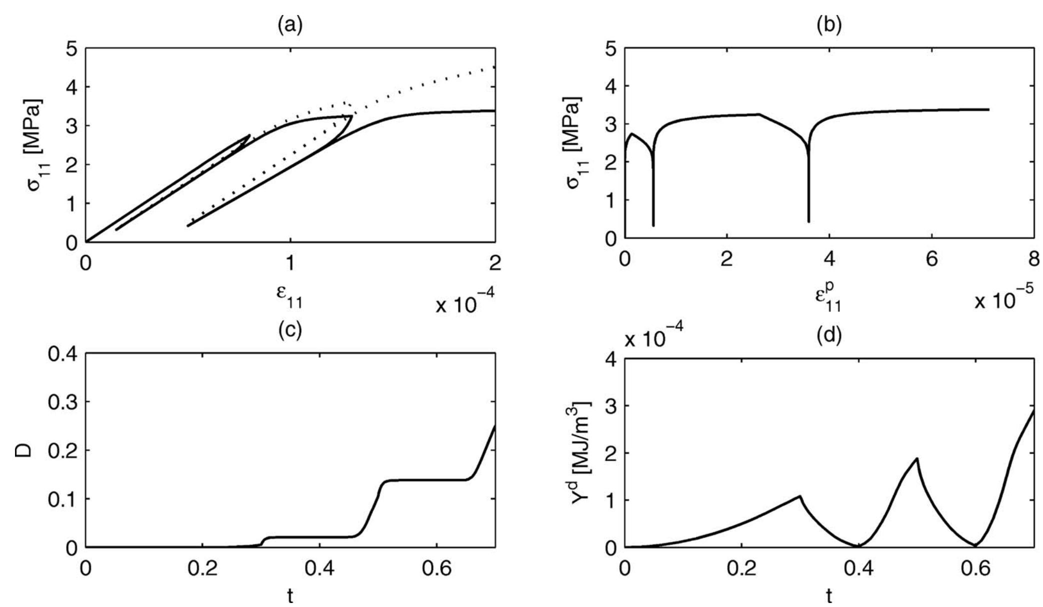

Fig. 6. Uniaxial behavior of an endochronic plasticity model with damage (Definition 2). (a) The stress–strain evolution. The dotted line indicates the effective stress $\sigma_{11}/(1-D)$. (b) The stress as function of the plastic strain. Observe that the plastic strain increases during unloading phases. (c) The damage evolution, with damage slightly increasing also during unloading phases. (d) Evolution of Y^d versus time.

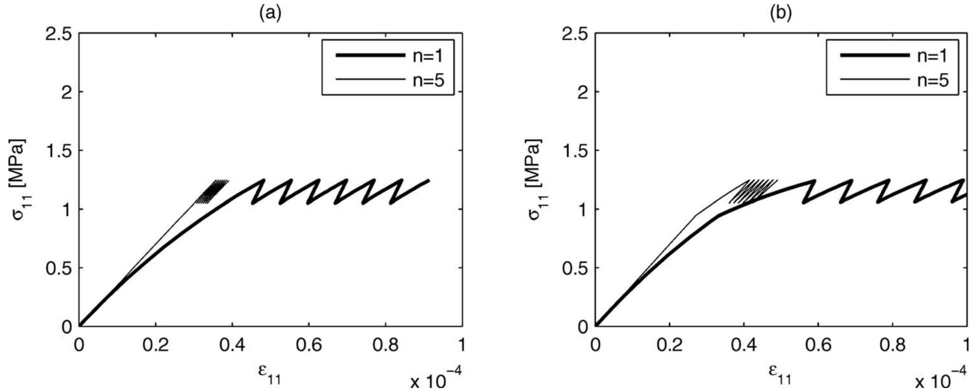

Fig. 7. Strain accumulation, uniaxial behavior with stress varying between 1.05 and 1.24 MPa. (a) Endochronic model without damage, with the intrinsic time (4). (b) Endochronic model with damage, with the intrinsic time (9) and the damage evolution given by Eqs. (27) and (33), with $r_0=0.000012$ and $s=6$.

Moreover, there are materials that are stable in the Drucker's sense and others that are not. Hence, for these materials, a proper model cannot fulfill the postulate of Drucker. All the aspects concerning this subject have been explored in detail in the previously cited references (Sandler 1978; Bažant 1978) for endochronic models without damage. A detailed analysis for the case of models with damage would deserve further studies, but this is beyond the purposes of this paper. Figs. 7(a) and 8(a) simply show the influence of the parameter n on the strain accumulation and the stress relaxation for an endochronic model without damage. When n tends to infinity, a plastic behavior of Prandtl-Reuss type is retrieved, where neither strain accumulation nor stress relaxation occur. Figs. 7(b) and 8(b) concern models with damage.

Another important topic concerning plasticity and/or damage models is the loss of uniqueness due to strain-softening (see, e.g., Jirásek and Bažant 2002). An exhaustive treatment of this subject for endochronic models with damage requires further analyses. However, for illustrative purposes, a simple analytical study of a uniaxial model is presented hereafter. Let σ, ε, and ε^p be the stress, the total strain and the plastic strain in the axial direction, respectively. Then, the uniaxial behavior can be represented by the following law: $\sigma=(1-D)E(\varepsilon-\varepsilon^p)=(1-D)E\varepsilon^e$, where E = Young modulus. The incremental form reads

$$d\sigma=(1-D)E(d\varepsilon-d\varepsilon^p)-\sigma\frac{dD}{1-D}$$

$$=(1-D)Ed\varepsilon-\beta\sigma\frac{d\zeta}{g}-\sigma\frac{dD}{1-D} \tag{41}$$

where the intrinsic time increment is

$$d\zeta=\left(1+\frac{\gamma}{\beta}\,\mathrm{sign}(\sigma d\varepsilon)\right)|\sigma|^{n-1}(1-D)^{1-n}|d\varepsilon| \tag{42}$$

and the damage increment writes

$$dD=H(f_D)\frac{dR}{d\varepsilon^e}d\varepsilon^e\frac{1-D}{sR(\varepsilon^e)} \tag{43}$$

with $R(\varepsilon^e)=E\langle\varepsilon^e\rangle^2/2$ and $f_D=(1-D)^sR(\varepsilon^e)-r_0\leqslant 0$. Assume $\sigma>0$ and $d\varepsilon>0$ (loading); the case $\sigma<0$, $d\varepsilon<0$ is analogous. Then, the condition to avoid strain-softening is

$$d\sigma\geqslant 0 \tag{44}$$

The generic damage increment when $f_D=0$ is given by

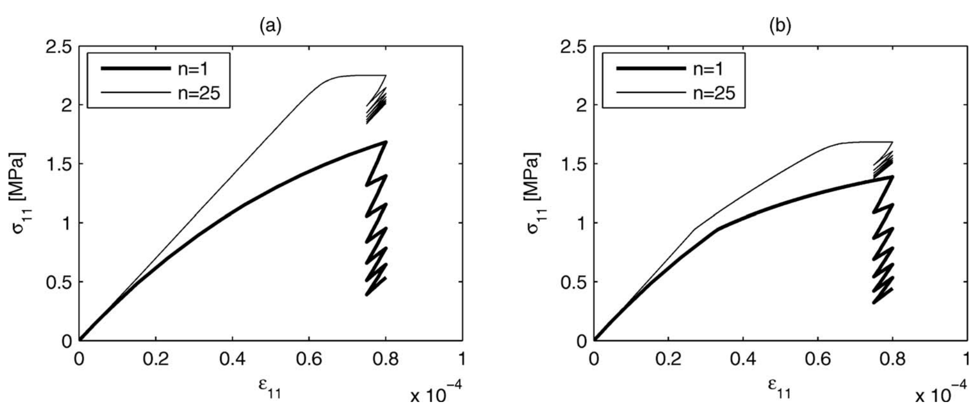

Fig. 8. Stress relaxation, uniaxial behavior with strain varying between 0.000075 and 0.00008. Endochronic model (a) without damage and (b) with damage, with the same damage parameters as in Fig. 7(b).

$$dD = \frac{2}{s}\frac{1-D}{\varepsilon^e}d\varepsilon^e = \frac{2}{s}E\frac{(1-D)^2}{\sigma}(d\varepsilon - d\varepsilon^p)$$

Moreover, from Eq. (41) one has $d\varepsilon^p = \beta\sigma d\zeta/[E(1-D)g]$. Hence, the condition (44) takes the following form:

$$\left((1-D)Ed\varepsilon - \beta\sigma\frac{d\zeta}{g}\right)\left(1-\frac{2}{s}\right) \geqslant 0 \qquad (45)$$

The first factor is always positive provided that $g \geqslant 1$. This can be proven using the definition of $d\zeta$ given in Eq. (42) with $\sigma > 0, d\varepsilon > 0$ and observing that the nonnegativity of the first factor in Eq. (45) is equivalent to the condition $\sigma/(1-D) \leqslant (E/(\beta+\gamma))^{1/n}(g)^{1/n} = \sigma_y(g)^{1/n}$, stating that the effective stress is always less or equal than the bounding axial stress σ_y, modified by the hardening function g. If $g \geqslant 1$, this inequality is always strictly fulfilled. Hence, strain softening can be avoided if $s \geqslant 2$. The same result can be obtained using the tensor expressions (7), (9), (27), and (33) and imposing that all the stress components are zero except $\sigma_{11} := \sigma$. This proof is omitted for brevity. The same condition on s has been found for the case of elasticity with damage (Nedjar 2001). Note that $g < 1$ induces strain softening also when there is no damage. The analysis of the unloading case is not necessary, as at a given stress–strain state with $\sigma \neq 0$, the unloading stiffness is always greater than the loading one. A more complex analysis, not considered here, is needed for the multiaxial case, where the fourth-order tensor of tangential moduli for the endochronic model with damage should be computed. If strain softening is avoided, the uniaxial behavior in what concerns the strain accumulation and the stress relaxation is analogous to that of standard endochronic models.

Conclusions

An extended endochronic theory with a scalar damage variable was developed, based on the postulate of strain equivalence and by using pseudopotentials depending on state variables and on parameters related to the past history of the material. The relevant loading surfaces, for damage and for plasticity, were defined. Two different damage pseudopotentials were discussed and a formalization of the conditions on state variables affecting the definition of damage was provided, by an additional indicator function in the Helmholtz free energy. In a companion paper (Erlicher and Bursi 2008), a link between this extended endochronic theory and the Bouc–Wen type models with both strength and stiffness degradation is established. This will permit one to prove the thermodynamic admissibility of these Bouc–Wen models and to highlight a constraint for the relevant stiffness degradation rules.

Notation

The following symbols are used in this paper:

$\mathbf{C}$ = fourth-order elasticity tensor;
D = internal variable associated with isotropic damage;
$\mathbb{D}$ = effective domain of the pseudopotential ϕ_p;
$\mathbb{D}_D$ = effective domain of the pseudopotential ϕ_D;
E = Young's modulus;
$\mathbb{E}$ = plasticity loading domain;
$\mathbb{E}_D$ = damage loading domain;
e = total dissipated energy per unit volume;
e_D = energy per unit volume dissipated through damage;
e_p = energy per unit volume dissipated through plasticity;
f_D = loading function for damage;
f_p = loading function for plasticity;
G = shear modulus;
g = hardening-softening function;
$H(\cdot)$ = Heaviside function;
$\mathbb{H}$ = effective domain of Ψ;
$\mathbf{I}$ = fourth-order identity tensor;
$\mathbb{I}_{\mathbb{H}}$ = indicator function of the set $\mathbb{H}$;
K = bulk modulus;
n = parameter related to the definition of the intrinsic time flow $\dot{\zeta}$;
$\mathbf{q}^d$ = dissipative thermodynamic forces vector;
$\mathbf{q}^{nd}$ = nondissipative thermodynamic forces vector;
$\mathbf{q}^{ndr}$ = nondissipative reaction thermodynamic forces vector;
R = function called *source of damage*;
R^d = dissipative thermodynamic force dual to ζ;
R^{nd} = nondissipative thermodynamic force dual to ζ;
R^{ndr} = nondissipative reaction thermodynamic force dual to ζ;
r_0 = initial damage threshold;
s = parameter related to the definition of the damage evolution;
$\mathbf{v}$ = state variables vector;
Y^d = dissipative thermodynamic force dual to the damage variable;
Y^{nd} = nondissipative thermodynamic force dual to the damage variable;
Y^{ndr} = nondissipative reaction thermodynamic force dual to the damage variable;
$\mathbf{z}$ = *hysteretic* part of the stress tensor;
β = coefficient related to the definition on the plastic flow $\dot{\boldsymbol{\varepsilon}}^p$;
γ = coefficient related to the definition of the intrinsic time flow $\dot{\zeta}$;
$\boldsymbol{\varepsilon}$ = total strain tensor;
$\boldsymbol{\varepsilon}^e$ = elastic strain tensor;
$\boldsymbol{\varepsilon}^{e,+}$ = positive part of $\boldsymbol{\varepsilon}^e$;
$\boldsymbol{\varepsilon}^p$ = plastic strain tensor;
ζ = intrinsic time *measure* for endochronic models. Moreover, it is the internal variable associated with isotropic hardening/softening of endochronic models;
ϑ = intrinsic time *scale* for endochronic models;
$\dot{\lambda}$ = plastic multiplier;
$\dot{\lambda}_D$ = damage multiplier;
μ = memory kernel;
ν = Poisson's ratio;
$\boldsymbol{\rho}$ = history-dependent parameters vector;
$\boldsymbol{\sigma}$ = Cauchy stress tensor;
$\boldsymbol{\sigma}^d$ = dissipative part of $\boldsymbol{\sigma}$;
$\boldsymbol{\sigma}^{nd}$ = nondissipative part of $\boldsymbol{\sigma}$;
$\boldsymbol{\sigma}^{ndr}$ = nondissipative reaction part of $\boldsymbol{\sigma}$;
σ_u = deviatoric bounding stress;
σ_y = axial bounding stress;
$\boldsymbol{\tau}^d$ = nondissipative thermodynamic force dual to the plastic strain;
$\boldsymbol{\tau}^{nd}$ = dissipative thermodynamic force dual to the plastic strain;
$\boldsymbol{\tau}^{ndr}$ = nondissipative reaction thermodynamic force dual to the plastic strain;
Φ_1 = mechanical or intrinsic dissipation;

ϕ = pseudopotential or dissipation potential;
ϕ_D = pseudopotential or dissipation potential for damage;
ϕ_p = pseudopotential or dissipation potential for plasticity;
ϕ_D^* = dual pseudopotential for damage;
ϕ_p^* = dual pseudopotential for plasticity;
Ψ = Helmholtz free energy volume density;
ψ = regular part of the Helmholtz free energy volume density;
$\mathbf{1}$ = second-order identity tensor; and
$\langle\ \rangle$ = McCauley brackets.

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