Controllability: finite dimension, heat equation

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Finite dimension controllability

1 Framework

Consider matrices $A : \mathbb{R}^N \to \mathbb{R}^N$, $B : \mathbb{R}^M \to \mathbb{R}^N$ (think $M << N$). Take some function $u : [0,T] \to \mathbb{R}^M$, $y_0 \in \mathbb{R}^N$, and consider $y : [0,T] \to \mathbb{R}^N$ the solution to

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, \ t \in [0,T], \\ y(0) = y_0. \end{cases} \quad (1)$$

Question: take $y_1 \in \mathbb{R}^N$, is there u such that $y(T) = y_1$?

2 Intuition

Consider an explicit scheme ($\tau > 0$): $\frac{d}{dt}y(t) \simeq \frac{y(t+\tau) - y(t)}{\tau}$ and put it in (1), $u_n := u(n\tau)$, $x_n := y(n\tau) \longrightarrow$

$$x_{n+1} = x_n + \tau(Ax_n + Bu_n) = (I + \tau A)x_n + \tau Bu_n. \tag{2}$$

Propagate (2), and take $N(= \dim \mathbb{R}^N)$ iterations from 0 gives

$$\begin{aligned} x_N = {} & \tau Bu_{N-1} + (I + \tau A)\tau Bu_{N-2} + \\ & .... + (I + \tau A)^{N-1}\tau Bu_0 + (Id + \tau A)^{N-1}\tau Ax_0. \end{aligned} \tag{3}$$

Form the matrix

$$[\tau B \;\; (I + \tau A)\tau B],$$

and see that

$$\mathrm{Im}([\tau B \;\; (I + \tau A)\tau B]) = \mathrm{Im}([B \;\; AB]),$$

see that

$$\mathrm{Im}([\tau B \;\; (I + \tau A)\tau B \;\; (I + \tau A)^2\tau B]) = \mathrm{Im}([B \;\; AB \;\; A^2B]),$$

....

$$\mathrm{Im}([\tau B \;\; (I+\tau A)\tau B \;\; (I+\tau A)^2\tau B \;\; \ldots \; (I+\tau A)^{N-1}\tau B]) =$$

$$\mathrm{Im}([B \;\; AB \;\; A^2B \;\; \ldots \; A^{N-1}B]).$$

Now if more than N steps of time, then x_N can be any y_1 if

$$\mathrm{Im}([B \;\; AB \;\; A^2B \;\; \ldots \; A^{N-1}B]) = \mathbb{R}^N.$$

Remark that this condition is ``necessary'' in the case of N iterations exactly.

3 Result for the ode

Theorem 1. Kalman's condition. The necessary and sufficient condition, such that, given y_0, for any y_1 there exists u such that $y(T) = y_1$ where y is the solution of (1) is that

$$\mathrm{Im}([B \;\; AB \;\; A^2B \;\; A^3B \;\ldots A^{N-1}B]) = \mathbb{R}^N. \qquad (4)$$

We speak then about exact controllability.

Remarks:

- The condition does not depend on T.
- The condition does not depend on y_0.
- The condition is extremely difficult to check.
- I did not precise the type of solutions of (1): if u is not regular, may not even be able to define a solution....

4 Examples

$$\begin{cases} \frac{d}{dt}x_1 = x_1 + x_2 + u \\ \frac{d}{dt}x_2 = x_1 + x_2 - u \end{cases} \text{ is not controllable.}$$

$$\begin{cases} \frac{d}{dt}x_1 = -x_2 \\ \frac{d}{dt}x_2 = -x_1 + u \end{cases} \text{ is controllable.}$$

$$\begin{cases} \frac{d}{dt}x_1 = -x_2 \\ \frac{d}{dt}x_2 = -x_1 + u \end{cases} \text{ is ????.}$$

5 Mild solution

Formally, how to find the solution of (1).

Apply the Duhamel's principle:

Find the general solution without $Bu(t)$:

$$y(t) = \exp(At)y_0$$

where $\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}$, then ``the'' solution of (1) is

$$y(t) = \exp(At)y_0 + \int_0^t \exp(A(t-s))Bu(s)ds. \quad (5)$$

If $u \in L^2((0,T), \mathbb{R}^M)$ then (5) leads to y which satisfies (1) almost everywhere.

6 Check the controllability condition on the mild solution

To obtain any y_1, it is clear that it is necessary and sufficient that the map

$$u \mapsto \int_0^T \exp(A(T-s))Bu(s)ds \in \mathbb{R}^N$$

is onto. In finite dimension it is the case iff one has $\forall u \in L^2((0,T),\mathbb{R}^N),\ \int_0^T {}^t v \exp(A(T-s))Bu(s)ds = 0 \Rightarrow v = 0$, for $v \in \mathbb{R}^N$.

□ Proof of the theorem:

Assume that

$$u \mapsto \int_0^T \exp(A(T-s))Bu(s)ds \in \mathbb{R}^N$$

is not onto.

Then there exists $v \neq 0$ such that for any u one has

$$\int_0^T {}^t v \exp(A(T-s))Bu(s)ds = 0.$$

Take $u(s) := {}^tB \exp({}^tA(T-s))v$.

Then $\int_0^T |{}^t v \exp(A(T-s))B|^2 ds = 0.$

Thus $\forall t \in [0,T]$, ${}^t v \exp(At)B = 0.$

But $t \mapsto {}^t v \exp(At) B$ is an analytic map, thus the coefficient of its Taylor expansion should be 0.

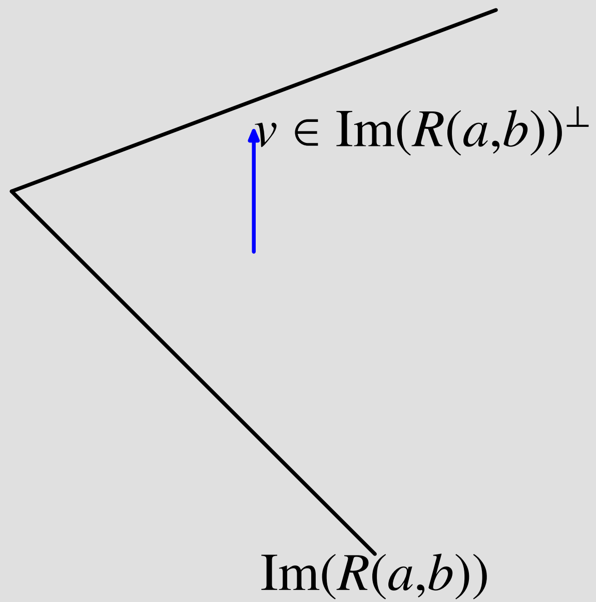
They are ${}^t vB$, ${}^t vAB$, ..., ${}^t vA^{N-1}B$, ... meaning that

$$R(A,B) := [B\ AB\ A^2B \ldots A^{N-1}B]$$

has an non zero orthogonal to its image and thus is not onto.

Orthogonality

Figure 1



- Conversely:

Let us denote $b_1, ..., b_M$ the column of B, the image of $R(A,B)$ is thus generated by $b_1, ..., A^{N-1}b_1$, $b_2, ..., A^{N-1}b_2, ..., b_M, ..., A^{N-1}b_M$.

But necessarily each of this group of N vectors generates a stable subspace of $\mathbb{R}^N$ by A.

For example, if $b_1, ..., A^{N-1}b_1$ is a free family then $A^N b_1 \in \text{Vect}(b_1, ..., A^{N-1}b_1)$, if not there exists a combination $A^k b_1 + \lambda_1 A^{k-1} b_1 + ... + \lambda_k b_1 = 0$ and we are done.

Thus, for any $P \in \mathbb{N}$, $A^P b_i \in Im(R(A,B))$.

If the image of $R(A,B)$ is not $\mathbb{R}^N$, there then exists a nonzero $v \in \mathbb{R}^N$, such that ${}^t v A^P b_i = 0$ for any $P \leq N-1$ and b_i, thus for any $P \in \mathbb{N}$. Thus ${}^t v \exp(tA)B = 0$ for any $t \in [0,T]$ which proves that the map

$$u \mapsto \int_0^T \exp(A(T-s))Bu(s)ds$$

is not onto.

7 Minimal control

Assume that the Kalman condition is satifisied, that is $R(A,B)$ has image $\mathbb{R}^N$. Then $\int_0^T \exp(A(T-s))Bu(s)ds$ can be anything with an appropriate u. But you can add to u anything in the kernel of $L : u \mapsto \int_0^T \exp(A(T-s))Bu(s)ds$.

So look for $u \in (\ker L)^\perp = \mathrm{Im} L^*$, where L^* is defined by $\langle L^* v | u \rangle_{L^2(0,T;\mathbb{R}^M)} = \langle v | Lu \rangle_{\mathbb{R}^N}$.
Easy computation

$$L^* v(s) := {}^t B \exp({}^t A (T - s)) v.$$

The control of minimal norm can be looked for with such a form and thus is regular. So a real solution of the equation (1).

8 Remark on exact controllability

We have proven the Kalman's condition by playing on the orthogonal of the image. It works only in finite dimension where the orthogonal of the image is 0 iff the image is the total space. In infinite dimension this is equivalent to a dense image.

9 What if the Kalman's condition fails

- Take $e_1, ..., e_r$ basis of $\mathrm{Im}(R(A,B))$. Complete with $e_{r+1}, .., e_N$ a basis of $\mathbb{R}^N$. P the matrix with component of $e_1, ..., e_N$. $B' = P^{-1}B$ $A' = P^{-1}AP$. Put $z = P^{-1}y$. Then y sol of (1) iff z sol of $\frac{d}{dt}z = A'z + B'u$.
- Now $B' = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ $A' = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, B_1 is $r \times M$, A_1 is $r \times r$, A_3 is $N-r \times N-r$. Write $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$\frac{d}{dt}z_1 = A_1 z_1 + A_2 z_2 + B_1 u_1$, controlled part $\frac{d}{dt}z_2 = A_3 z_3$, uncontrolled part, and $R(A_1,B_1)$ has rank r.

Extensions

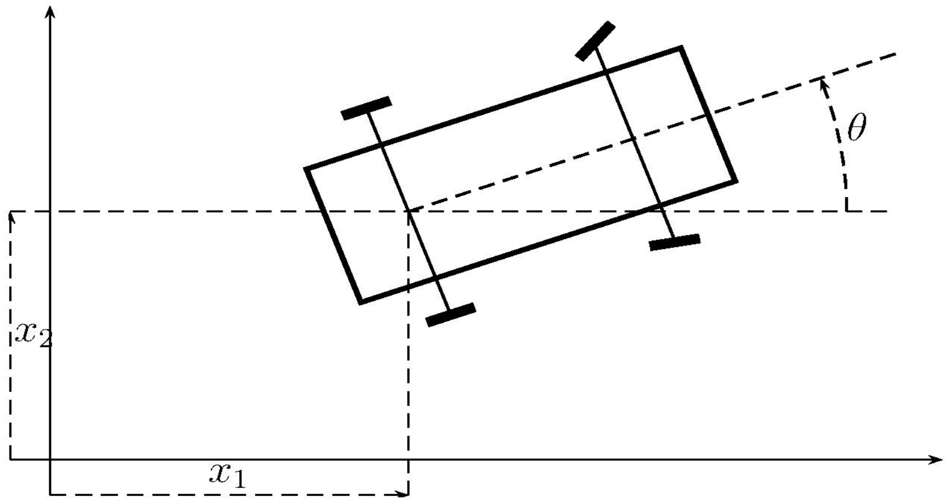
Consider more general cases of control problem even in finite dimension: *e.g*

$$\frac{d}{dt}y(t) = u_1(t)X_1(y(t)) + u_2(t)X_2(y(t))$$

where X_1 X_2 have value in $\mathbb{R}^3$, and u_1 u_2 are real functions. Presumably only possible to go in the direction X_1 or X_2.
But X_1 et X_2 depend on the position. Thus their variation may lead to other directions:

The baby stroller

The baby stroller: The model



$$\dot{x}_1 = u_1 \cos x_3,\ \dot{x}_2 = u_1 \sin x_3, \dot{x}_3 = u_2,\ n = 3,\ m = 2.$$

Theorem 2. Chow's theorem.Everything is C^∞. If $\forall y$, vect$(X_1, X_2, [X_1, X_2]) = \mathbb{R}^3$ then initial y_0 can be driven to anything. With some u_1 and u_2 with $\frac{d}{dt}y(t) = u_1(t)X_1(y(t)) + u_2(t)X_2(y(t))$.

$[X_1, X_2]$ (Lie bracket) is a way of measuring some defects in the Schwarz's theorem ($\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$).

But Chow's theorem (and some of its generalizations) has great consequences:

Entropy (Caratheodory) principle

Let a system be governed by independant variables $(V_1, V_2, V_3) \in \mathbb{R}^3$

Assume that $\forall(V_1,V_2,V_3)$, $\exists A(V_1,V_2,V_3)$ a subspace that characterizes some admissible paths; i.e. $t \mapsto \gamma(t)$ has velocity $\gamma'(t) \in A(\gamma(t))$.
This A has a varying basis (X_1,X_2).
A generalization of Chow's result (due to H. Sussmann) allows to make a dichotomy.

- Either all iterated lie brackets generates $\mathbb{R}^3$ then any pair of (V_1,V_2,V_3) can be joined by admissible path.
- Either there is at least a pair of (V_1,V_2,V_3) that cannot be joined by some admissible paths.

In the last situation, if for some reasons, you can characterize the $A(V_1, V_2, V_3)$ by the kernel of a linear form $\delta Q(V_1, V_2, V_3) : \mathbb{R}^3 \to \mathbb{R}$ where δQ is regular enough,
Now by the Frobenius theorem (generalization of Cauchy-Lipschitz) the violation of Chow's theorem $\Rightarrow$ there exist from any point a surface tangent to A that one characterizes by some relation $S(V_1, V_2, V_3) = 0$, $S : \mathbb{R}^3 \to \mathbb{R}$. Then by construction $\ker dS = \ker \delta Q$, thus, there should be some function T such that

$$dS = \frac{\delta Q}{T}.$$

Of course δQ is the exchange of heat, T is the temperature, S the entropy which characterizes the irreversibility of the Joule's effects.

Figure 2 A. Y. hand

Irreversible path with respect to pressure and temperature. Application of entropy under that framework to economy may be found in

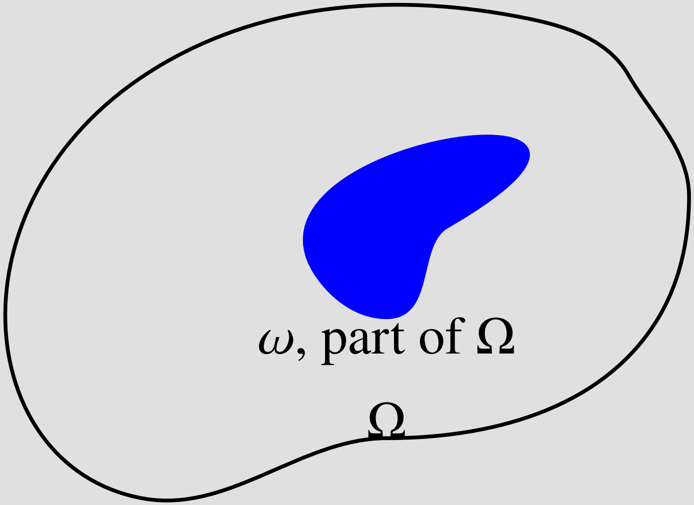
Geogescu-Roegen: La décroissance. Entropie, écologie, économie.

The heat equation with a distributed control h

10 Framework

Open regular set Ω

Figure 3



Controlled heat equation

$$\begin{cases} \partial_t y - \Delta y = u \text{ in } (0,T) \times \Omega, \\ y = 0 \text{ on } (0,T) \times \partial\Omega \\ y(t=0) = y_0, \text{ the initial data.} \end{cases} \quad (6)$$

11 Notion of solutions

- Strong solutions of (6) :

$$y_0 \in L^2(\Omega),\ u \in C((0,T), H^2(\Omega) \cap H_0^1(\Omega)),$$

or

$$u \in W^{1,1}((0,T), L^2(\Omega)).$$

- Weak solutions of (6) : on the account of the properties of the Laplace operator $A = -\Delta$ on $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, one can consider $u \in L^2(0,T,L^2(\Omega))$ and define weak solutions: satisfy pde of (6) in some (distribution) sense, and satisfy the boundary conditions in another (weak) sense: basically to be able to satisfy some formal integration by parts.
If $y_0 \in L^2(\Omega)$, there exists $C > 0$ universal such that
$$|y|_{L^2((0,T)\times\Omega)} \leq C(|y_0|_{L^2(\Omega)} + |u|_{L^2((0,T)\times\Omega)}).$$

12 Distributed control

One considers $u \in L^2((0,T) \times \omega)$, that is the control acts only in ω, and is 0 elsewhere.

□ Natural question: Given $y_1 \in L^2(\Omega)$ can one find $u \in L^2((0,T)\times\omega)$ such that y the solution of (6) satisfies $y(T,.) = y_1$?

If so for any y_0 and y_1, exact (distributed on ω) controllability for the heat equation.

But

$$\forall t \in (0,T],\ y(t,.) \in C^\infty(\Omega \setminus \omega).$$

Thus not exact (distributed) controllability.

- Natural question again: Given $y_1 \in L^2(\Omega)$, and $\varepsilon > 0$, can one find $u \in L^2((0,T) \times \Omega)$ such that y the solution of (6) satisfies $|y(T,.) - y_1|_0 < \varepsilon$?
If so for any y_0 and y_1, approximate (distributed on ω) controllability for the heat equation.

This time, Bingo.

Theorem 3.

- Natural question again: What are the y_1 that can be attained ?

Open question......

☐ Take $h = 0$, any $\widetilde{y_0} \in L^2(\Omega)$ and compute $\widetilde{y}$ the corresponding solution of (6) with $h = 0$ and compute $\widetilde{y}(T)$.
Given y_0, can one find u such that the corresponding solution of (6) satisfies $y(T) = \widetilde{y}(T)$?
If so for any y_0 and any $\widehat{y_0}$, then exact controllability to trajectories.

Bingo again.

Theorem 4.

In fact equivalent to $\forall y_0$ find u such that $y(T) = 0$ (exact zero controllability). (Exercise)

Formalism of controllability

- Rewrite (6) like a system

$$\begin{cases} \frac{d}{dt}y = Ay + Bu \\ y(0) = y_0 \end{cases} \tag{7}$$

Express formally the solution of (7) with the Duhamel principle in the semi-group form

$$y(t) = S(t)y_0 + \int_0^t S(t-s)Bu(s)ds, \tag{8}$$

where $A : D(A) = H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is $-\Delta$ and $B : L^2((0,T) \times \omega) \to L^2((0,T) \times \Omega)$ is the extension by 0.
One sees on (8) that exact controllability to trajectories occurs iff exact zero controllability occurs.

To ascertain exact zero controllability, find u such that

$$\int_0^T S(T-s)Bu(s)ds = -S(T)y_0.$$

- The question of exact zero controllability is then formulated as:

Let $\mathcal{L} : L^2((0,T)\times\Omega) \to L^2(\Omega)$

$$\mathcal{L}(u) := \int_0^T S(T-s)Bu(s)ds,$$

does there hold $\mathcal{L}(L^2((0,T)\times\Omega)) \supset S(T)(L^2(\Omega))$?

Abstract framework

Let H_1, H_2, H_3 Hilbert spaces, $C_3 : D(C_3) \subset \overline{D(C_3)} = H_3 \to H_1$ a closed operator, $C_2 : H_2 \to H_1$ a continuous linear operator, then it is equivalent to say

$$\exists M \geq 0,\ \forall u \in D(C_3^*),\ |C_2^* u|_{H_2} \leq M |C_3^* u|_{H_3}$$

or to say

$$C_2(H_2) \subset C_3(D(C_3)),$$

and if so, there exists $C_1 : H_2 \to D(C_3)$ such that

$$|C_1|_{L(H_2,H_3)} \leq M,\ C_2 = C_3 C_1.$$

Proof:

□ If $C_2(H_2) \subset C_3(D(C_3))$, for any $z_2 \in H_2$ $\exists z_3 \in D(C_3)$ such that $C_2(z_2) = C_3(z_3)$. Take $z_3 \in \ker(C_3)^\perp$ and put $C_1(z_2) = z_3$.

- Let us show that C_1 is continuous, or that C_1 satisfies the closed graph property. Take a converging sequence z_{2n} of H_2, and assume that $C_1(z_{2n})$ is convergent. By continuity of C_2, $C_2(z_{2n})$ is convergent, and thus $C_3(C_1(z_{2n}))$ is convergent, thus, since C_3 is closed (meaning here its graph is closed), $C_1(z_{2n})$ is converging to some element in $\ker(C_3)^\perp$, which can only be $C_1(\lim z_{2n})$, thus the continuity.

- Let $M := |C_1|_{L(H_2,H_3)}$. Then $\forall z_1 \in D(C_3^*)$

$$|C_2^* z_1|_{H_2} = \max_{z_2 \in H_2,\ |z_2|_{H_2}=1} \langle C_2^* z_1 | z_2 \rangle_{H_2} = \max_{z_2 \in H_2,\ |z_2|_{H_2}=1} \langle z_1 | C_3 C_1 z_2 \rangle_{H_1}$$

$$= \max_{z_2 \in H_2,\ |z_2|_{H_2}=1} \langle C_3^* z_1 | C_1 z_2 \rangle_{H_2} \leq M |C_3^* z_1|_{H_2}$$

□ Conversely, if $\forall u \in D(C_3^*)$, $|C_2^* u|_{H_2} \leq M |C_3^* u|_{H_3}$.
Let $A : C_3^*(D(C_3^*)) \to H_2$ defined by

$$A(C_3^*(u)) = C_2^* u.$$

A is basically (linear) continuous on continuous $C_3^*(D(C_3^*)) \longrightarrow$ extends uniquely to $\overline{C_3^*(D(C_3^*))}$, and by 0 to the orthogonal.
Take $C_1 = A^*$....

Back to the heat equation

- Take $C_2 = S(T)$ with $H_2 := L^2(\Omega)$ and $C_3 : L^2((0,T) \times \omega) \to L^2(\Omega)$ given by

$$L(u) := \int_0^T S(T-t)Bu(s)ds.$$

Exact zero controllability for the heat equation $\iff$ there exists $M > 0$ such that

$$\forall z \in L^2(\Omega),\ |S(T)^* z|_{L^2(\Omega)} \leq M |L^*(z)|_{L^2((0,T)\times\omega)}.$$

Equality known as observability inequality. Very difficult to obtain.

Computation of $S(T)^*$

Take $u = 0$ in 6.

For ϕ in $L^2(\Omega)$, formally (Φ defined on $[0,T] \times \Omega$ with $\Phi(T) = \phi$.)

$$\int_\Omega \phi y(T) dx$$
$$= \int_0^T \int_\Omega \partial_t \Phi y dx dt + \int_0^T \int_\Omega \partial_t y \Phi dx dt + \int_\Omega \Phi(0) y(0) dx$$
$$= \int_0^T \int_\Omega \partial_t \Phi y dx dt + \int_0^T \int_\Omega \Delta \Phi y dx dt + \int_\Omega \Phi(0) y(0) dx,$$

if we assume $\Phi = 0$ on $(0,T) \times \Omega$.

Take then Φ the solution to (the well-defined problem)

$$\begin{cases} \partial_t \Phi + \Delta \Phi = 0 \text{ on } (0,T) \times \Omega \\ \Phi = 0 \text{ on } (0,T) \times \partial\Omega \\ \Phi(T) = \phi, \end{cases} \tag{9}$$

then

$$S(T)^*(\phi) = \Phi(0).$$

Computation of L^*:

Exercice

$$L^* \phi = \Phi_{|(0,T)\times\omega}.$$

Observability inequality

Find some $M \geq 0$ such that $\forall \phi \in L^2(\Omega)$ the solution of (9) satisfies

$$|\Phi(0)|^2_{L^2(\Omega)} \leq M^2 \int_\omega \int_0^T |\Phi|^2 dxdt.$$

Observe that if Φ is 0 on $(0,T) \times \omega$ then $\Phi(0) = 0$ then $\phi = 0$. Because: take $e_n \in H^1_0(\Omega)$ an eigenvector of $-\Delta$ with eigenvalue λ_n, that is $-\Delta e_n = \lambda_n e_n$. Take also $\leq_n |_0 = 1$. $0 < \lambda_{n-1} \leq \lambda_n \to \infty$.

Then if $\phi = \sum_{n=0}^\infty \mu_n e_n$, then $\Phi(t) = \sum_{n=0}^\infty e^{\lambda_n (t-T)} \mu_n e_n$. Thus $\Phi(0) = 0 \Rightarrow \mu_n = 0, \forall n \in \mathbb{N}$.

The observability inequality is true for any T, and can be obtained by means of Carleman's inqualities.

Unlike modern fashion nowadays Carleman was a very productive mathematician, but published very few papers.....

Many results and pictures taken from the books of
J.M. Coron.
E. Sontag.
J. Zabczyk.

Efkaristo poli for your attention.